

# GLOBAL SOLUTION TO THE NEMATIC LIQUID CRYSTAL FLOWS WITH HEAT EFFECT

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**ABSTRACT.** The temperature-dependent incompressible nematic liquid crystal flows in a bounded domain  $\Omega \subset \mathbb{R}^N$  ( $N = 2, 3$ ) are studied in this paper. Following Danchin's method in [J. Math. Fluid Mech., 2006], we use a localization argument to recover the maximal regularity of Stokes equation with variable viscosity, by which we first prove the local existence of strong solution, then extend it to a global one provided that the initial data is a sufficiently small perturbation around the trivial equilibrium state. This paper also generalizes Hu-Wang's result in [Commun. Math. Phys., 2010] to the non-isothermal case.

## 1. INTRODUCTION

Liquid crystal is an intermediate state of matter between isotropic fluids and crystalline solids. Such materials can be artificially obtained typically by increasing the temperature of a solid crystal (low molecular weight) or increasing the concentration of certain solvent (high molecular weight). Among various types of liquid crystals, *nematic* ( $\nu\eta\mu\alpha$ , thread) ones are those composed of rod-like molecules with head-to-tail symmetry. For more physical and chemical background on the underlying subject, we refer to [6, 15] and the references therein.

In this paper, we will focus on the mathematical analysis on the following hydrodynamic system, which is a macroscopic continuum description of the time evolution of homogeneous non-isothermal incompressible nematic liquid crystals.

$$(1.1) \quad \begin{cases} \partial_t u + u \cdot \nabla u - \operatorname{div} (\mu(\theta) \mathcal{D}(u)) + \nabla P = -\Delta d \cdot \nabla d, \\ \partial_t d + u \cdot \nabla d = \Delta d + |\nabla d|^2 d, \\ \partial_t \theta + u \cdot \nabla \theta - \Delta \theta = \frac{1}{2} \mu(\theta) |\mathcal{D}(u)|^2 + |\Delta d + |\nabla d|^2 d|^2, \\ \operatorname{div} u = 0, \quad |d| = 1. \end{cases}$$

The above equations correspond to conservation of linear momentum, angular momentum, internal energy, incompressibility and physical constraint on the director fields. Here, we denote by  $u$ ,  $d$ ,  $P$  and  $\theta$  the velocity, director, pressure and temperature, respectively.  $\mathcal{D}(u) = \frac{1}{2}(\nabla u + \nabla^T u)$  is the Cauchy stress tensor. The total energy density of the system is  $e := \frac{1}{2}(|u|^2 + |\nabla d|^2) + \theta$ . System (1.1) is a simplified version of those proposed in [35, 37].

Suppose the nematic liquid crystals are confined in a bounded domain  $\Omega \subset \mathbb{R}^N$  ( $N = 2, 3$ ), the following initial-boundary conditions are imposed in this paper.

$$(1.2) \quad (u, d, \theta)|_{t=0} = (u_0, d_0, \theta_0), \quad (u, \mathcal{B}d, \partial_\nu \theta)|_{\partial\Omega} = (0, 0, 0),$$

where  $\mathcal{B}d = \partial_\nu d$  or  $d - \mathbf{e}$ ,  $\mathbf{e}$  is a fixed unit constant vector and  $\nu$  is the outward normal vector on  $\partial\Omega$ . Moreover,  $\operatorname{div} u_0 = 0$ ,  $|d_0| = 1$ ,  $d_0|_{\partial\Omega} = \mathbf{e}$ . One easily checks that if  $(u, d, P, \theta)$  is a smooth solution to (1.1), then under the initial-boundary conditions (1.2), the total energy is conserved along with the time evolution:

$$\frac{d}{dt} \int_{\Omega} e(t, x) dx = 0.$$

If neglecting the heat effect and  $\mu$  is constant, then (1.1) reduces to the simplified version of the so-called Ericksen-Leslie system which is developed by Ericksen, Leslie et. al [10, 11, 25, 26] in the 1960s. For such simplified system, Lin [29] and Lin-Liu [30, 31] initiated the study on the Ginzburg-Landau approximated system in 1990s. Specifically, they replaced  $|\nabla d|^2 d$  by a penalty function  $1/\epsilon^2(1 - |d_\epsilon|^2)d_\epsilon$  and the physical constraint  $|d| = 1$  is relaxed. For fixed  $\epsilon$ , they obtained the global well-posedness and partial regularity of the approximated system in two and three dimensional space.

As for the analysis of the original simplified system, it is more challenging. However, there has been some important results in the two dimensional case, thanks to the local *a priori* estimates on  $\Delta d$  obtained due to a Ladyzhenskja-type inequality by Struwe [38]. In 2010, Lin-Lin-Wang [32] proved the existence of global weak solutions, which is regular except for possible finite time slices. At the same time, Hong [19] obtained the same results by proving the convergence of the solutions to the approximated system as  $\epsilon \rightarrow 0_+$ . Similar results have been achieved for more general (stress tensor) systems by Hong-Xin [21], Huang-Lin-Wang [23] and Wang-Wang [40]. The uniqueness of the above weak solutions is also proved by Lin-Wang [22], Xu-Zhang [41] and Li-Titi-Xin [27]. Also it's worth remarking that Lei-Li-Zhang [24] generalized Ding-Lin's results on the harmonic maps flow in [9], proved that if the initial director satisfies a natural angle condition, then the weak solutions obtained in [19, 32] are actually smooth.

For the three dimensional case, the approach by Ladyzhenskja-type inequality fails, little is known for global well-posedness under general large initial data, except that Lin-Wang [34] proved the existence of global weak solution if the initial director is targeted on a hemisphere. As for local well-posedness, there are some results by Hong-Li-Xin [20] for the Oseen-Frank model. Also for small perturbations around trivial equilibrium state  $(0, \mathbf{e})$ , global strong solution is proved in [16], through a quasilinear approach. And Wang [39] obtained the global mild solutions for initial data  $(u_0, d_0)$  belonging to possibly the largest space  $BMO^{-1} \times BMO$ , with small norm.

If heat effect is considered, the system is energetically closed but more complicated. Feireisl-Rocca-Schimperna [13] and Feireisl-Fremond-Rocca-Schimperna [12] first studied the approximated system with heat effect and obtained the existence of the global weak solution in two and three dimensional space. Later, Li-Xin [28] proved the existence of global weak solutions to system (1.1) in  $\mathbb{R}^2$ . The uniqueness and regularity of such weak solutions remain open.

As mentioned in the beginning, low-molecular-weight nematic liquid crystals generally are sensitive to the variation of temperature, especially near the threshold of phase transitions. According to the physical experiment in [7], the principal viscosity of nematic liquid crystals is a continuously differentiable function of temperature

in the nematic phase. In particular, we may assume that

$$(1.3) \quad 0 < \underline{\mu} \leq \mu(\theta) \leq \bar{\mu} < \infty, \quad |\mu'(\theta)| \leq \bar{\mu}' < \infty \quad \forall \theta,$$

where  $\underline{\mu}$ ,  $\bar{\mu}$  and  $\bar{\mu}'$  are material constants. According to the physical experiments, nematic liquid crystals usually display some instability near the phase transition thresholds, while in the nematic phase, it is generally expected to be stable. In this paper, we give a rigorous mathematical proof for such stability, at least for model (1.1). Notice that  $(0, \mathbf{e}, \theta_*)$  is always a trivial equilibrium state to (1.1), where  $\theta_*$  is arbitrary constant. Without loss of generality, we will set  $\theta_*$  to be 0 from now on. And we will prove that there always exists a unique global strong solution to (1.1) if the initial data is a sufficiently small perturbation of this trivial equilibrium state. We would like to remark that recently Hieber-Prüss proved the stability of  $(0, \mathbf{e}, 1)$  for a more general model, via a quasilinear approach in [17]. Also the assumption on the viscosity is different from ours. Here a linear approach is adopted, which the authors also consider to be of independent interest.

Before stating our main result, we set up the functional spaces for strong solutions. Here a *strong solution* on  $\Omega_T$  means that a set  $(u, d, P, \theta)$  satisfying system (1.1) almost everywhere with initial boundary conditions (1.2) and belongs to  $E_T^{p,q,r,s}$ , which is defined as the following.

**Definition 1.1.** For  $T > 0$  and  $1 \leq p, q, r, s < \infty$ , we denote  $E_T^{p,q,r,s}$  by the set of  $(u, d, P, \theta)$  such that

$$\begin{aligned} u &\in C([0, T]; D_{A_r}^{1-\frac{1}{p}, p}(\Omega)) \cap W^{1,p}(0, T; L^r(\Omega)) \cap L^p(0, T; W^{2,r}(\Omega)), \\ d &\in C([0, T]; B_{r,p}^{3-\frac{2}{p}}(\Omega)) \cap W^{1,p}(0, T; W^{1,r}(\Omega)) \cap L^p(0, T; W^{3,r}(\Omega)), \\ P &\in W^{1,p}(0, T; W^{1,r}(\Omega)), \quad \int_{\Omega} P \, dx = 0, \\ \theta &\in C([0, T]; B_{q,s}^{2-\frac{2}{s}}(\Omega)) \cap W^{1,s}(0, T; L^q(\Omega)) \cap L^s(0, T; W^{2,q}(\Omega)). \end{aligned}$$

Obviously  $E_T^{p,q,r,s}$  is a Banach space, we denote its natural norm as  $\|\cdot\|_{E_T^{p,q,r,s}}$ . We also remark that the condition

$$\int_{\Omega} P \, dx = 0$$

in the above definition holds automatically if we replace  $P$  by  $P - \frac{1}{|\Omega|} \int_{\Omega} P \, dx$  in system (1.1). Also

$$D_{A_r}^{1-\frac{1}{p}, p} := (L_{\sigma}^r, D(A_r))_{1-\frac{1}{p}, p},$$

where

$$L_{\sigma}^r(\Omega) := \{u \in L^r(\Omega), \operatorname{div} u = 0\}, \quad D(A_r) = \{u \in W^{2,r}(\Omega), \operatorname{div} u = 0, u|_{\partial\Omega} = 0\}.$$

Moreover, it follows from Proposition 2.5 in [8] that

$$D_{A_r}^{1-\frac{1}{p}, p} \hookrightarrow B_{r,p}^{2(1-\frac{1}{p})} \cap L_{\sigma}^r(\Omega).$$

The Besov space  $B_{r,p}^{2(1-\frac{1}{p})}$  on a bounded domain can be regarded as the interpolation space between  $L^r$  and  $W^{2,r}$ , that is,

$$B_{r,p}^{2(1-\frac{1}{p})} = (L^r, W^{2,r})_{1-\frac{1}{p}, p}.$$

We note that this kind of strong solution has been proved to exist for the density-dependent incompressible Navier-Stokes equations by Danchin [8] and the simplified Ericksen-Leslie system without the term  $|\nabla d|^2 d$  by Hu-Wang [22]. Inspired by their work, we generalize the above results to system (1.1). Our first result on the local existence is as follows:

**Theorem 1.2.** *Suppose  $\Omega \subset \mathbb{R}^N (N = 2, 3)$  is a bounded domain with smooth boundary. (1.3) holds true,  $u_0 \in D_{A_r}^{1-\frac{1}{p}, p}$ ,  $d_0 \in B_{r,p}^{3-\frac{2}{p}}$  and  $\theta_0 \in B_{q,s}^{2-\frac{2}{s}}$  with  $p, q, r$  and  $s$  satisfying*

$$1 < p < \infty, \quad 2 \leq s < \infty, \quad N < r \leq q, \quad \frac{2}{p} + \frac{N}{r} < \frac{1}{s} + \frac{N}{2q} + 1 < 2.$$

*Then there exists  $T_0 > 0$  such that the system (1.1)–(1.2) has a unique local strong solution  $(u, d, P, \theta) \in E_{T_0}^{p,q,r,s}$ ,  $T_0$  depends on the initial data.*

Our second result is on the global existence of strong solutions.

**Theorem 1.3.** *Under the conditions of Theorem 1.2, in addition, assume that  $p \leq 2s$ , then there exists  $\delta > 0$  such that if*

$$\|(u_0, d_0 - \mathbf{e}, \theta_0)\|_{D_{A_r}^{1-\frac{1}{p}, p} \times B_{r,p}^{3-\frac{2}{p}} \times B_{q,s}^{2-\frac{2}{s}}} \leq \delta,$$

*then system (1.1)–(1.2) admits a unique global strong solution  $(u, d, P, \theta) \in E_T^{p,q,r,s}$ , for any  $T > 0$  and*

$$(1.4) \quad \|(u, d - \mathbf{e}, P, \theta)\|_{E_\infty^{p,q,r,s}} \leq C\delta,$$

*for some  $C$  independent of initial data and time.*

*Remark 1.4.* Concerning the above two theorems, we make following remarks.

- (1) In the above two theorems, the index set is not empty. In fact, one admissible choice for both is  $s = p = 2$  and  $q = r > N$ . Here we have one more restriction  $p \leq 2s$  for global existence is due to some time independent interpolation inequality in Lemma 3.3.
- (2) It is interesting to compare our result to those in [18], in which an example of finite time blow-up is given for arbitrarily small  $\|u_0\|_{L^2} + \|\nabla d_0\|_{L^2}$ . There is no contradiction here since our smallness condition is stronger.
- (3) Notice that Theorem 1.3 actually proves the stability of equilibrium state in nematic phase,  $(0, \mathbf{e}, 0)$ . But whether it is asymptotically stable or not is unknown to the authors at this moment, we hope to address this issue in a future paper.

The rest of this paper is organized as follows. In Section 2, we first state the maximal regularity for linear parabolic equation and Stokes equation with variable viscosity. In Section 3, we establish the local existence and uniqueness of strong solution to system (1.1) by an iteration method. In Section 4, the global existence of strong solution for small perturbations around the trivial equilibrium state is proved. Finally, in the appendix, the proof of maximal regularity of Stokes equation with variable viscosity is presented.

## 2. THE LINEAR ESTIMATES

**2.1. Linear parabolic equation.** First we recall the maximal regularity for the parabolic operators (cf. Theorem 4.10.7 and Remark 4.10.9 in [4]):

**Theorem 2.1.** *Given  $1 < p, q < \infty$ , for the Cauchy problem*

$$(2.1) \quad \begin{cases} \partial_t \omega - \Delta \omega = f, \\ \omega(0) = \omega_0, \end{cases}$$

i) if  $\omega_0 \in B_{q,p}^{2-\frac{2}{p}}$  and  $f \in L^p(0, T; L^q(\mathbb{R}^N))$ , then system (2.1) has a unique solution  $\omega \in W^{1,p}(0, T; L^q) \cap L^p(0, T; W^{2,q})$  satisfying

$$(2.2) \quad \begin{aligned} & \|\omega\|_{C([0,T]; B_{q,p}^{2-\frac{2}{p}})} + \|\omega\|_{W^{1,p}(0,T; L^q) \cap L^p(0,T; W^{2,q})} \\ & \leq C(\|\omega_0\|_{B_{q,p}^{2-\frac{2}{p}}} + \|f\|_{L^p(0,T; L^q)}), \end{aligned}$$

where  $C$  is independent of  $\omega_0$ ,  $f$  and  $T$ .

ii) if  $\omega_0 \in B_{q,p}^{3-\frac{2}{p}}$  and  $f \in L^p(0, T; W^{1,q}(\mathbb{R}^N))$ , then system (2.1) has a unique solution  $\omega \in W^{1,p}(0, T; W^{1,q}) \cap L^p(0, T; W^{3,q})$  satisfying

$$(2.3) \quad \begin{aligned} & \|\omega\|_{C([0,T]; B_{q,p}^{3-\frac{2}{p}})} + \|\omega\|_{W^{1,p}(0,T; W^{1,q}) \cap L^p(0,T; W^{3,q})} \\ & \leq C(\|\omega_0\|_{B_{q,p}^{3-\frac{2}{p}}} + \|f\|_{L^p(0,T; W^{1,q})}), \end{aligned}$$

where  $C$  is independent of  $\omega_0$ ,  $f$  and  $T$ .

*Remark 2.2.* Notice that the above results also hold true for the Neumann or Dirichlet problem on bounded domain with sufficiently regular boundary.

**2.2. Linearized Stokes equation.** The following theorem plays a key role in our analysis.

**Theorem 2.3.** *Let  $\Omega \subset \mathbb{R}^N$  be a bounded domain with smooth boundary,  $1 < p, r < \infty$ ,  $N < r \leq q$  and  $\mu$  satisfies (1.3).  $u_0 \in D_{A_r}^{1-\frac{1}{p}, p}$ ,  $f \in L^p(0, T; L^r)$  and  $\theta$  satisfies*

$$(2.4) \quad \theta \in L^\infty(0, T; W^{1,q}) \cap \dot{C}^\beta(0, T; L^\infty),$$

for some  $\beta \in (0, 1)$ . Then the system

$$(2.5) \quad \begin{cases} \partial_t u - \operatorname{div}(\mu(\theta)\mathcal{D}(u)) + \nabla P = f, \\ \operatorname{div} u = 0, \quad \int_\Omega P \, dx = 0, \\ u|_{t=0} = u_0, \quad u|_{\partial\Omega} = 0, \end{cases}$$

has a unique solution  $(u, P)$  satisfying

$$(2.6) \quad \begin{aligned} & \|u\|_{C([0,t]; D_{A_r}^{1-\frac{1}{p}, p})} + \|(\partial_t u, \Delta u, \nabla P)\|_{L_t^p(L^r)} \\ & \leq C B_\theta^k(t) e^{C t C_\theta(t)} (\|u_0\|_{D_{A_r}^{1-\frac{1}{p}, p}} + \|f\|_{L_t^p(L^r)}), \end{aligned}$$

and

$$(2.7) \quad \begin{aligned} & \|u\|_{C([0,t]; D_{A_r}^{1-\frac{1}{p}, p})} + \|(\partial_t u, \Delta u, u, \nabla P)\|_{L^p(0,t; L^r)} \\ & \leq C \left( B_\theta^k(t) (\|u_0\|_{D_{A_r}^{1-\frac{1}{p}, p}} + \|f\|_{L^p(0,t; L^r)}) + C_\theta(t) \|u\|_{L^p(0,t; L^r)} \right), \end{aligned}$$

for any  $0 < t \leq T$ , where  $C$  is independent of  $u_0$ ,  $f$ ,  $\theta$  and  $T$ ,

$$B_\theta(t) = 1 + \|\nabla \theta\|_{L_t^\infty(L^q(\Omega))}^{\frac{q}{q-N}}, \quad C_\theta(t) = B_\theta^{l_1}(t)(\|\nabla \theta\|_{L_t^\infty(L^q)} + \|\theta\|_{\dot{C}^\beta(0,t;L^\infty)})^{l_2},$$

$k, l_1 \geq 2, l_2 \geq 1$  depending only on  $p, q, r$  and  $s$ .

*Remark 2.4.* As shown in [8],  $r > N$  is actually not necessary, we here impose this condition to simplify the index, also this is the case we need in proving our main results. Also as one shall see in the proof,  $k \geq 2$  and  $k \rightarrow \infty$  as  $q \rightarrow N$ , so this estimates does not work for the critical case. Finally notice that  $l_2 > 1$  plays a crucial role in the proof of global existence of strong solutions.

The solvability for variable viscosity Stokes equation is well-know in principle, for example we refer to [1, 2, 5, 36]. Here in order to close the estimates, we need to derive the estimates with explicit dependence on the viscosity. Also for the completeness of our presentation, we give an independent proof in the appendix.

### 3. EXISTENCE ON A SMALL TIME INTERVAL

Before we proceed, some interpolation inequalities are introduced as preparation. Such inequalities can be also found in [8, 22].

**Lemma 3.1.** *Under the conditions of Theorem 1.2, it holds that*

$$(3.1) \quad \|\nabla f\|_{L_T^p(L^\infty)} \leq CT^{\frac{1}{2}(1-\frac{N}{r})} \|f\|_{L_T^\infty(B_{r,p}^{2-\frac{2}{p}})}^{\frac{p}{2}(1-\frac{N}{r})} \|f\|_{L_T^p(W^{2,r})}^{1-\frac{p}{2}(1-\frac{N}{r})},$$

$$(3.2) \quad \|\nabla f\|_{L_T^{2s}(L^{2q})} \leq CT^{\frac{1}{2}(1-\frac{2}{p}+\frac{1}{s}-\frac{N}{r}+\frac{N}{2q})} \|f\|_{L_T^\infty(B_{r,p}^{2-\frac{2}{p}})}^{\frac{p}{2}(1-\frac{N}{r}+\frac{N}{2q})} \|f\|_{L_T^p(W^{2,r})}^{1-\frac{p}{2}(1-\frac{N}{r}+\frac{N}{2q})},$$

$$(3.3) \quad \|\nabla f\|_{L_T^{4s}(L^{4q})} \leq CT^{\frac{1}{2}(2-\frac{2}{p}+\frac{1}{2s}-\frac{N}{r}+\frac{N}{4q})} \|f\|_{L_T^\infty(B_{r,p}^{3-\frac{2}{p}})}^{\frac{p}{2}(2-\frac{N}{r}+\frac{N}{4q})} \|f\|_{L_T^p(W^{3,r})}^{1-\frac{p}{2}(2-\frac{N}{r}+\frac{N}{4q})},$$

where  $C$  depends only on  $p, q, r, s$  and  $\Omega$ .

*Proof.* The proof of the lemma is mainly based on the interpolation and Hölder inequality. Noticing that

$$B_{r,p}^{2-\frac{2}{p}} \hookrightarrow B_{\infty,\infty}^{2-\frac{2}{p}-\frac{N}{r}}, \quad W^{2,r} \hookrightarrow B_{\infty,\infty}^{2-\frac{N}{r}}, \\ (B_{\infty,\infty}^{1-\frac{2}{p}-\frac{N}{r}}, B_{\infty,\infty}^{1-\frac{N}{r}})_{\gamma,1} = B_{\infty,1}^{s_0} \hookrightarrow L^\infty,$$

where  $1 - \frac{N}{r} - \frac{2}{p}(1-\gamma) = s_0 \geq 0$ , for some  $\gamma \in [0, 1)$ , then it follows that

$$\begin{aligned} \|\nabla f\|_{L_T^p(L^\infty)} &\leq C \left( \int_0^T \|\nabla f\|_{B_{\infty,1}^{s_0}}^p dt \right)^{\frac{1}{p}} \\ &\leq C \left( \int_0^T \|\nabla f\|_{B_{\infty,\infty}^{1-\frac{2}{p}-\frac{N}{r}}}^{p(1-\gamma)} \|\nabla f\|_{B_{\infty,\infty}^{1-\frac{N}{r}}}^{p\gamma} dt \right)^{\frac{1}{p}} \\ &\leq CT^{\frac{1-\gamma}{p}} \|f\|_{L_T^\infty(B_{r,p}^{2-\frac{2}{p}})}^{1-\gamma} \|f\|_{L_T^p(W^{2,r})}^\gamma. \end{aligned}$$

Therefore, (3.1) is proved.

Similarly, notice that  $B_{r,p}^{2-\frac{2}{p}} \hookrightarrow B_{2q,p}^{2-\frac{2}{p}-\frac{N}{r}+\frac{N}{2q}}$  ( $2q > r$ ) and  $W^{2,r} \hookrightarrow B_{r,r}^2 \hookrightarrow B_{2q,r}^{2-\frac{N}{r}+\frac{N}{2q}}$  ( $r > N \geq 2$ ). On the other hand,

$$(B_{2q,p}^{1-\frac{2}{p}-\frac{N}{r}+\frac{N}{2q}}, B_{2q,r}^{1-\frac{N}{r}+\frac{N}{2q}})_{\gamma,1} = B_{2q,1}^{s_0} \hookrightarrow L^{2q},$$

where  $1 - \frac{N}{r} + \frac{N}{2q} - \frac{2}{p}(1 - \gamma) = s_0 \geq 0$ , for some  $\gamma \in [0, 1]$ . Consequently,

$$\begin{aligned}
\|\nabla f\|_{L_T^{2s}(L^{2q})} &\leq C \left( \int_0^T \|\nabla f\|_{B_{2q,1}^{s_0}}^{2s} dt \right)^{\frac{1}{2s}} \\
&\leq C \left( \int_0^T \|f\|_{B_{r,p}^{2-\frac{2}{p}}}^{2s(1-\gamma)} \|f\|_{W^{2,r}}^{2s\gamma} dt \right)^{\frac{1}{2s}} \\
&\leq C \|f\|_{L_T^\infty(B_{r,p}^{2-\frac{2}{p}})}^{1-\gamma} \left( \int_0^T \|f\|_{W^{2,r}}^{2s\gamma \cdot \frac{p}{2s\gamma}} dt \right)^{\frac{1}{2s} \cdot \frac{2s\gamma}{p}} T^{\frac{1}{2s}(1-\frac{2s\gamma}{p})} \\
&\leq CT^{\frac{1}{2s}(1-\frac{2s\gamma}{p})} \|f\|_{L_T^\infty(B_{r,p}^{2-\frac{2}{p}})}^{1-\gamma} \|f\|_{L_T^p(W^{2,r})}^\gamma.
\end{aligned}$$

In the above computation, Hölder inequality is applicable since  $p > 2s\gamma$ , which is guaranteed by our condition in the theorem.

Finally, notice that  $B_{r,p}^{3-\frac{2}{p}} \hookrightarrow B_{4q,p}^{3-\frac{2}{p}-\frac{N}{r}+\frac{N}{4q}}$ ,  $W^{3,r} \hookrightarrow B_{4q,r}^{3-\frac{N}{r}+\frac{N}{4q}}$  and

$$(B_{4q,p}^{2-\frac{2}{p}-\frac{N}{r}+\frac{N}{4q}}, B_{4q,r}^{2-\frac{N}{r}+\frac{N}{4q}})_{\gamma,1} = B_{4q,1}^{s_0} \hookrightarrow L^{4q},$$

where  $2 - \frac{N}{r} + \frac{N}{4q} - \frac{2}{p}(1 - \gamma) = s_0 \geq 0$ , for some  $\gamma \in [0, 1]$ . Thus,

$$\begin{aligned}
\|\nabla f\|_{L_T^{4s}(L^{4q})} &\leq C \left( \int_0^T \|\nabla f\|_{B_{4q,1}^{s_0}}^{4s} dt \right)^{\frac{1}{4s}} \\
&\leq C \left( \int_0^T \|f\|_{B_{r,p}^{3-\frac{2}{p}}}^{4s(1-\gamma)} \|f\|_{W^{3,r}}^{4s\gamma} dt \right)^{\frac{1}{4s}} \\
&\leq CT^{\frac{1}{4s}(1-\frac{4s\gamma}{p})} \|f\|_{L_T^\infty(B_{r,p}^{3-\frac{2}{p}})}^{1-\gamma} \|f\|_{L_T^p(W^{3,r})}^\gamma,
\end{aligned}$$

where we have used  $4s\gamma < p$ , in other words,  $0 \leq \frac{2}{p} - \frac{1}{2s} + \frac{N}{r} - \frac{N}{4q} < 2$ , and this is the direct consequence of our condition.  $\square$

*Remark 3.2.* Since  $D_{A_r}^{1-\frac{1}{p},p} \hookrightarrow B_{r,p}^{2-\frac{2}{p}}$ , the space  $B_{r,p}^{2-\frac{2}{p}}$  in (3.1) and (3.2) could be replaced by  $D_{A_r}^{1-\frac{1}{p},p}$  accordingly.

The above lemma is mainly used to deal with the nonlinear terms of the system in the process of proving the local strong solution for general initial data. However, in the global estimates, some uniform in time control on the nonlinear terms is needed. To this end, we also have the following version of interpolation inequality:

**Lemma 3.3.** *Under the conditions of Theorem 1.3, it holds that*

$$(3.4) \quad \|\nabla f\|_{L_T^p(L^\infty)} \leq C \|f\|_{L_T^p(W^{2,r})},$$

$$(3.5) \quad \|\nabla f\|_{L_T^{2s}(L^{2q})} \leq C \|f\|_{L_T^\infty(B_{r,p}^{2-\frac{2}{p}})}^{1-\frac{p}{2s}} \|f\|_{L_T^p(W^{2,r})}^{\frac{p}{2s}},$$

$$(3.6) \quad \|\nabla f\|_{L_T^{4s}(L^{4q})} \leq C \|f\|_{L_T^\infty(B_{r,p}^{3-\frac{2}{p}})}^{1-\frac{p}{4s}} \|f\|_{L_T^p(W^{3,r})}^{\frac{p}{4s}},$$

where  $C$  depends only on  $p, q, r, s$  and  $\Omega$ .

*Proof.* (3.10) immediately follows from the fact that  $W^{2,r} \hookrightarrow W^{1,\infty}$  as  $r > N$ . Secondly, since  $p \leq 2s$ , by the log-convexity of  $L^p$  norms (for example see page 27 in [3]),

$$\|\nabla f\|_{L_T^{2s}(L^{2q})} \leq \|\nabla f\|_{L_T^\infty(L^{2q})}^{1-\frac{p}{2s}} \|\nabla f\|_{L_T^p(L^{2q})}^{\frac{p}{2s}} \leq C \|f\|_{L_T^\infty(B_{r,p}^{2-\frac{2}{p}})}^{1-\frac{p}{2s}} \|f\|_{L_T^p(W^{2,r})}^{\frac{p}{2s}},$$

where we have used the fact that  $B_{r,p}^{2-\frac{2}{p}} \hookrightarrow W^{1,2q}$  as  $\frac{2}{p} + \frac{N}{r} < 1 + \frac{N}{2q}$  and  $W^{2,r} \hookrightarrow W^{1,2q}$  as  $r > N$ , so (3.5) is proved. By the same token, one can easily check (3.6).  $\square$

Next, we begin to prove the existence of local strong solution through an iteration method. The proof will be divided into the following steps.

**Step 1: Construction of approximate solution.** We initialize the construction of approximate solution by setting  $u^0 := u_0$ ,  $d^0 := d_0$  and  $\theta^0 := \theta_0$ . Given  $(u^n, d^n, P^n, \theta^n) \in E_T^{p,q,r,s}$  for some  $T > 0$ , Theorem 2.1 enables us to define  $\theta^{n+1}$  as the unique solution of the system

$$(3.7) \quad \begin{cases} \partial_t \theta^{n+1} - \Delta \theta^{n+1} \\ \quad = -u^n \cdot \nabla \theta^n + \frac{1}{2} \mu(\theta^n) |\mathcal{D}(u^n)|^2 + |\Delta \bar{d}^n + |\nabla \bar{d}^n|^2 (\bar{d}^n + \mathbf{e})|^2, \\ \theta^{n+1}|_{t=0} = \theta_0, \quad \partial_\nu \theta^{n+1}|_{\partial\Omega} = 0, \end{cases}$$

on  $\Omega_T$ , where  $\bar{d}^n := d^n - \mathbf{e}$ . Then by Theorem 2.1, define  $\bar{d}^{n+1}$  as the unique solution of system

$$(3.8) \quad \begin{cases} \partial_t \bar{d}^{n+1} - \Delta \bar{d}^{n+1} = -u^n \cdot \nabla \bar{d}^n + |\nabla \bar{d}^n|^2 (\bar{d}^n + \mathbf{e}), \\ \bar{d}^{n+1}|_{t=0} = d_0 - \mathbf{e}, \quad \bar{d}^{n+1}|_{\partial\Omega} = 0 \quad \text{or} \quad \partial_\nu \bar{d}^{n+1}|_{\partial\Omega} = 0. \end{cases}$$

Finally, Theorem 2.3 and (1.3) enables us to define  $(u^{n+1}, P^{n+1})$  by  $(u^n, d^n, \theta^{n+1})$  as the unique global solution of

$$(3.9) \quad \begin{cases} \partial_t u^{n+1} - \operatorname{div}(\mu(\theta^{n+1}) \mathcal{D}(u^{n+1})) + \nabla P^{n+1} = -u^n \cdot \nabla u^n - \Delta \bar{d}^n \cdot \nabla \bar{d}^n, \\ \operatorname{div} u^{n+1} = 0, \quad \int_\Omega P^{n+1} dx = 0, \\ u^{n+1}|_{t=0} = u_0, \quad u^{n+1}|_{\partial\Omega} = 0. \end{cases}$$

Also Theorem 2.1 and 2.3 yield that  $(u^{n+1}, \bar{d}^{n+1}, P^{n+1}, \theta^{n+1}) \in E_T^{p,q,r,s}$ .

**Step 2: Uniform bounds for some small fixed time  $T$ .** In this step, we aim at finding a positive time  $T$  independent of  $n$  for which  $(u^n, \bar{d}^n, P^n, \theta^n)_{n \in \mathbb{N}}$  is uniformly bounded in the Banach space  $E_T^{p,q,r,s}$ .

In order to keep our presentation brief, let us denote

$$\begin{aligned} U_n(t) &:= \|u^n\|_{L_t^\infty(B_{A_r}^{1-\frac{1}{p},p})} + \|u^n\|_{W_t^{1,p}(L^r) \cap L_t^p(W^{2,r})} + \|P^n\|_{L_t^p(W^{1,r})}, \\ D_n(t) &:= \|\bar{d}^n\|_{L_t^\infty(B_{r,p}^{3-\frac{2}{p}})} + \|\bar{d}^n\|_{W_t^{1,p}(W^{1,r}) \cap L_t^p(W^{3,r})}, \\ \Theta_n(t) &:= \|\theta^n\|_{L_t^\infty(B_{q,s}^{2-\frac{2}{s}})} + \|\theta^n\|_{W_t^{1,s}(L^q) \cap L_t^s(W^{2,q})}, \\ U_0 &:= \|u_0\|_{D_{A_r}^{1-\frac{1}{p},p}}, \quad D_0 := \|d_0 - \mathbf{e}\|_{B_{r,p}^{3-\frac{2}{p}}}, \quad \Theta_0 := \|\theta_0\|_{B_{q,s}^{2-\frac{2}{s}}}, \\ E_n(t) &:= U_n(t) + D_n(t), \quad E_0 := U_0 + D_0. \end{aligned}$$



Then by Theorem 2.1, it holds that

$$(3.10) \quad \begin{aligned} \Theta_{n+1}(t) \leq & C \left( \Theta_0 + \underbrace{\|u^n \cdot \nabla \theta^n\|_{L_t^s(L^q)}}_{I_1} + \underbrace{\| |\nabla u^n|^2 \|_{L_t^s(L^q)}}_{I_2} \right. \\ & \left. + \underbrace{\| |\Delta \bar{d}^n + |\nabla \bar{d}^n|^2 (\bar{d}^n + \mathbf{e})|^2 \|_{L_t^s(L^q)}}_{I_3} \right), \end{aligned}$$

for any  $t > 0$ . Next, we evaluate the terms on the RHS of (3.10) one by one. It is noted that in the following the constant  $\gamma \in [0, 1]$  may vary in different inequalities and its value does not play a role in our analysis, thus from now on we do not distinguish them in notation unless otherwise claimed.

The first term on the RHS of (3.10) can be estimated as

$$(3.11) \quad \begin{aligned} I_1 & \leq C \|u^n\|_{L_t^\infty(L^q)} \|\nabla \theta^n\|_{L_t^s(L^\infty)} \\ & \leq C t^{\frac{1}{2}(1-\frac{N}{q})} \|u^n\|_{L_t^\infty(D_{A_r}^{1-\frac{1}{p},p})} \|\theta^n\|_{L_t^\infty(B_{q,s}^{2-\frac{2}{s}})}^{1-\gamma} \|\theta^n\|_{L_t^s(W^{2,q})}^\gamma \\ & \leq C t^{\frac{1}{2}(1-\frac{N}{q})} U_n(t) \Theta_n(t), \end{aligned}$$

where we have used the fact  $D_{A_r}^{1-\frac{1}{p},p} \hookrightarrow L^q$  as  $\frac{2}{p} + \frac{N}{r} < 2 + \frac{N}{q}$  and inequality (3.1).

For  $I_2$ , it follows from (3.2) that

$$(3.12) \quad \begin{aligned} I_2 & \leq C \|\nabla u^n\|_{L_t^{2s}(L^{2q})}^2 \\ & \leq C t^{1-\frac{2}{p}+\frac{1}{s}-\frac{N}{r}+\frac{N}{2q}} \|u^n\|_{L_t^\infty(D_{A_r}^{1-\frac{1}{p},p})}^{2(1-\gamma)} \|u^n\|_{L_t^p(W^{2,r})}^{2\gamma} \\ & \leq C t^{1-\frac{2}{p}+\frac{1}{s}-\frac{N}{r}+\frac{N}{2q}} U_n^2(t). \end{aligned}$$

Next, we evaluate the last term as

$$\begin{aligned} I_3 & \leq C (\|\Delta \bar{d}^n\|_{L_t^{2s}(L^{2q})}^2 + \| |\nabla \bar{d}^n|^2 (\bar{d}^n + \mathbf{e}) \|_{L_t^{2s}(L^{2q})}^2) \\ & \leq C (\|\Delta \bar{d}^n\|_{L_t^{2s}(L^{2q})}^2 + \|\nabla \bar{d}^n\|_{L_t^{4s}(L^{4q})}^4 (\|\bar{d}^n\|_{L_t^\infty(L^\infty)}^2 + 1)). \end{aligned}$$

Since

$$\begin{aligned} \|\Delta \bar{d}^n\|_{L_t^{2s}(L^{2q})} & \leq C t^{\frac{1}{2}(1-\frac{2}{p}+\frac{1}{s}-\frac{N}{r}+\frac{N}{2q})} \|\nabla \bar{d}^n\|_{L_t^\infty(B_{r,p}^{2-\frac{2}{p}})}^{1-\gamma} \|\nabla \bar{d}^n\|_{L_t^p(W^{2,r})}^\gamma \\ & \leq C t^{\frac{1}{2}(1-\frac{2}{p}+\frac{1}{s}-\frac{N}{r}+\frac{N}{2q})} D_n(t), \\ \|\nabla \bar{d}^n\|_{L_t^{4s}(L^{4q})} & \leq C t^{\frac{1}{2}(2-\frac{2}{p}+\frac{1}{2s}-\frac{N}{r}+\frac{N}{4q})} \|\bar{d}^n\|_{L_t^\infty(B_{r,p}^{3-\frac{2}{p}})}^{1-\gamma} \|\bar{d}^n\|_{L_t^p(W^{3,r})}^\gamma \\ & \leq C t^{\frac{1}{2}(2-\frac{2}{p}+\frac{1}{2s}-\frac{N}{r}+\frac{N}{4q})} D_n(t), \\ \|\bar{d}^n\|_{L^\infty(L^\infty)} & \leq C \|\bar{d}^n\|_{L^\infty(B_{r,p}^{3-\frac{2}{p}})} \leq C D_n(t), \end{aligned}$$

where we have used inequality (3.3) and  $B_{r,p}^{3-\frac{2}{p}} \hookrightarrow L^\infty$  as  $\frac{2}{p} + \frac{N}{r} < 3$ . Thus,

$$(3.13) \quad I_3 \leq C t^{1-\frac{2}{p}+\frac{1}{s}-\frac{N}{r}+\frac{N}{2q}} D_n^2(t) + C t^{2(2-\frac{2}{p}+\frac{1}{2s}-\frac{N}{r}+\frac{N}{4q})} D_n^4(t) (1 + D_n^2(t)).$$

Substituting (3.11), (3.12) and (3.13) into (3.10) yields that

$$(3.14) \quad \Theta_{n+1}(t) \leq C (\Theta_0 + t^{\xi_1} (\Theta_n^2(t) + E_n^2(t) + E_n^6(t))),$$

where  $\xi_1 = \min\{\frac{1}{2}(1-\frac{N}{q}), 1-\frac{2}{p}+\frac{1}{s}-\frac{N}{r}+\frac{N}{2q}, 2(2-\frac{2}{p}+\frac{1}{2s}-\frac{N}{r}+\frac{N}{4q})\} > 0$ .

Next, we move on to evaluate  $D_{n+1}(t)$ . Applying Theorem 2.1 to (3.8), one obtains

$$(3.15) \quad D_{n+1}(t) \leq C \left( D_0 + \underbrace{\|u^n \cdot \nabla \bar{d}^n\|_{L_t^p(W^{1,r})}}_{II_1} + \underbrace{\| |\nabla \bar{d}^n|^2 (\bar{d}^n + \mathbf{e}) \|_{L_t^p(W^{1,r})}}_{II_2} \right).$$

For the first term on the RHS of (3.15),

$$(3.16) \quad \begin{aligned} II_1 &\leq \|u^n \cdot \nabla \bar{d}^n\|_{L_t^p(L^r)} + \|\nabla u^n \cdot \nabla \bar{d}^n\|_{L_t^p(L^r)} + \|u^n \cdot \nabla^2 \bar{d}^n\|_{L_t^p(L^r)} \\ &\leq C(\|u^n\|_{L_t^\infty(L^r)} \|\nabla \bar{d}^n\|_{L_t^p(W^{1,\infty})} + \|\nabla u^n\|_{L_t^p(L^\infty)} \|\nabla \bar{d}^n\|_{L_t^\infty(L^r)}) \\ &\leq C t^{\frac{1}{2}(1-\frac{N}{r})} U_n(t) D_n(t). \end{aligned}$$

For the last term on the RHS of (3.15), by (3.4), one can obtain

$$(3.17) \quad \begin{aligned} II_2 &\leq \| |\nabla \bar{d}^n|^2 (\bar{d}^n + \mathbf{e}) \|_{L_t^p(L^r)} + 2 \|\nabla^2 \bar{d}^n \nabla \bar{d}^n (\bar{d}^n + \mathbf{e})\|_{L_t^p(L^r)} + \| |\nabla \bar{d}^n|^3 \|_{L_t^p(L^r)} \\ &\leq C(1 + \|\bar{d}^n\|_{L_t^\infty(W^{1,\infty})}) \|\nabla \bar{d}^n\|_{L_t^\infty(L^r)} \|\nabla \bar{d}^n\|_{L_t^p(W^{1,\infty})} \\ &\leq C t^{\frac{1}{2}(1-\frac{N}{r})} D_n^2(t) (1 + D_n(t)). \end{aligned}$$

Therefore, substituting (3.16) and (3.17) into (3.15) gives that

$$(3.18) \quad D_{n+1}(t) \leq C(D_0 + t^{\xi_2} (E_n^2(t) + E_n^3(t))),$$

for  $\xi_2 = \frac{1}{2}(1 - \frac{N}{r})$ .

Finally, applying Theorem 2.3 to (3.9), we get

$$(3.19) \quad \begin{aligned} U_{n+1}(t) &\leq C(1 + \|\nabla \theta^{n+1}\|_{L_t^\infty(L^q)})^\varsigma \exp(Ct(1 + \|\theta^{n+1}\|_{\dot{C}_t^\beta(L^\infty)} + \|\nabla \theta\|_{L_t^\infty(L^q)})^\varsigma) \\ &\quad \times (U_0 + \|u^n \cdot \nabla u^n\|_{L_t^p(L^r)} + \|\Delta \bar{d}^n \cdot \nabla \bar{d}^n\|_{L_t^p(L^r)}), \end{aligned}$$

for some  $\beta \in (0, 1)$  and  $1 < \varsigma < \infty$  depending on  $p, q, r$  and  $s$ .

Since  $s \geq 2$  and  $q > N$ , by Sobolev embedding,

$$W_t^{1,s}(L^q) \cap L^s(W^{2,q}) \hookrightarrow L_t^\infty(W^{1,q}) \cap \dot{C}_t^\beta(L^\infty), \quad \beta = 1 - \frac{1}{s} - \frac{N}{2q} \in (0, 1).$$

Thus, (3.19) reduces to

$$(3.20) \quad \begin{aligned} U_{n+1}(t) &\leq C(1 + \Theta_{n+1}(t))^\varsigma \exp(Ct(1 + \Theta_{n+1}(t))^\varsigma) \\ &\quad \times \left( U_0 + \underbrace{\|u^n \cdot \nabla u^n\|_{L_t^p(L^r)}}_{III_1} + \underbrace{\|\Delta \bar{d}^n \cdot \nabla \bar{d}^n\|_{L_t^p(L^r)}}_{III_2} \right). \end{aligned}$$

By the interpolation inequality (3.1), it follows that

$$(3.21) \quad \begin{aligned} III_1 &\leq C \|u^n\|_{L_t^\infty(L^r)} \|\nabla u^n\|_{L_t^p(L^\infty)} \\ &\leq C t^{\frac{1}{2}(1-\frac{N}{r})} \|u^n\|_{L_t^\infty(D_{Ar}^{1-\frac{1}{p},p})}^{2-\gamma} \|u^n\|_{L_t^p(W^{2,r})}^\gamma \\ &\leq C t^{\frac{1}{2}(1-\frac{N}{r})} U_n^2(t), \end{aligned}$$

$$(3.22) \quad \begin{aligned} III_2 &\leq C \|\Delta \bar{d}^n\|_{L_t^p(L^\infty)} \|\nabla \bar{d}^n\|_{L_t^\infty(L^r)} \\ &\leq C t^{\frac{1}{2}(1-\frac{N}{r})} \|\bar{d}^n\|_{L_t^\infty(B_{r,p}^{3-\frac{2}{p}})}^{2-\gamma} \|\bar{d}^n\|_{L_t^p(W^{3,r})}^\gamma \\ &\leq C t^{\frac{1}{2}(1-\frac{N}{r})} D_n^2(t). \end{aligned}$$

Substituting (3.21) and (3.22) into (3.20), one reaches

$$(3.23) \quad U_{n+1}(t) \leq C(1 + \Theta_{n+1}(t))^\varsigma \exp(Ct(1 + \Theta_{n+1}(t))^\varsigma) (U_0 + t^{\xi_2} E_n^2(t)).$$

Adding up (3.18) and (3.23), one infers that

$$(3.24) \quad E_{n+1}(t) \leq C(1 + \Theta_{n+1}(t))^\varsigma \exp(Ct(1 + \Theta_{n+1}(t))^\varsigma) (E_0 + t^{\xi_2} (E_n^3(t) + E_n^2(t))).$$

Assume that for some  $T > 0$  such that for any  $t \in [0, T]$ ,

$$\Theta_n(t) \leq CM_1(\Theta_0 + E_0), \quad E_n(t) \leq CM_2(\Theta_0 + E_0),$$

where  $M_1$  and  $M_2$  are some constants independent of  $T$  and to be determined later.

Choosing  $0 < T_1 \leq T$  such that

$$(3.25) \quad T_1^{\xi_1} (M_1^2 C^2 (\Theta_0 + E_0) + M_2^2 C^2 (\Theta_0 + E_0) + M_2^6 C^6 (\Theta_0 + E_0)^5) \leq 1,$$

then for any  $t \in [0, T_1]$ , it follows from (3.14) that

$$(3.26) \quad \Theta_{n+1}(t) \leq 2C(\Theta_0 + E_0).$$

Choosing  $0 < T_2 \leq T_1$  such that

$$(3.27) \quad \begin{cases} CT_2(1 + 2C(\Theta_0 + E_0))^\varsigma \leq \ln 2, \\ T_2^{\xi_2} (M_2^2 C^2 (\Theta_0 + E_0) + M_2^3 C^3 (\Theta_0 + E_0)^2) \leq 1, \end{cases}$$

then for any  $t \in [0, T_2]$ ,

$$(3.28) \quad E_{n+1}(t) \leq 4C(1 + 2C(\Theta_0 + E_0))^\varsigma (\Theta_0 + E_0).$$

Now choosing

$$M_1 = 2, \quad M_2 = 4(1 + 2C(\Theta_0 + E_0))^\varsigma,$$

thus one has

$$\Theta_{n+1}(t) \leq CM_1(\Theta_0 + E_0), \quad E_{n+1}(t) \leq CM_2(\Theta_0 + E_0), \quad \forall t \in [0, T_2]$$

Then by the induction argument,  $(u^n, d^n, P^n, \theta^n)$  is uniformly bounded in  $E_{T_2}^{p,q,r,s}$  with respect to  $n$ .

**Step 3: Convergence of sequence in  $E_T^{p,q,r,s}$  for some  $T < T_2$ .** In this step, we are devoted to proving that  $(u^n, d^n, P^n, \theta^n)$  is a Cauchy sequence in the Banach space  $E_T^{p,q,r,s}$  for sufficiently small  $T < T_2$ .

Let  $\delta u^n := u^{n+1} - u^n$ ,  $\delta P^n := P^{n+1} - P^n$ ,  $\delta d^n := d^{n+1} - d^n$ ,  $\delta \theta^n := \theta^{n+1} - \theta^n$ ,  $\delta \mu^n := \mu(\theta^{n+1}) - \mu(\theta^n)$ ,  $\delta \Theta_n := \Theta_{n+1} - \Theta_n$  and  $\delta E_n := E_{n+1} - E_n$ . Then  $(\delta u^n, \delta d^n, \delta P^n, \delta \theta^n)$  satisfies the system

$$(3.29) \quad \begin{cases} \partial_t \delta u^n - \operatorname{div}(\mu(\theta^{n+1}) \mathcal{D}(\delta u^n)) + \nabla \delta P^n \\ \quad = \operatorname{div}(\delta \mu^n \mathcal{D}(u^n)) - \delta u^{n-1} \cdot \nabla u^n - u^{n-1} \cdot \nabla \delta u^{n-1} \\ \quad \quad - \Delta d^n \cdot \nabla \delta d^{n-1} - \Delta \delta d^{n-1} \cdot \nabla d^{n-1}, \\ \partial_t \delta d^n - \Delta \delta d^n = |\nabla d^n|^2 \delta d^{n-1} + \nabla d^n \nabla \delta d^{n-1} d^{n-1} \\ \quad + \nabla \delta d^{n-1} \nabla d^{n-1} d^{n-1} - \delta u^{n-1} \cdot \nabla d^n - u^{n-1} \cdot \nabla \delta d^{n-1}, \\ \partial_t \delta \theta^n - \Delta \delta \theta^n = -u^n \cdot \nabla \delta \theta^{n-1} - \delta u^{n-1} \cdot \nabla \theta^{n-1} \\ \quad + \frac{1}{2} \delta \mu^{n-1} |\mathcal{D}(u^n)|^2 + \frac{1}{2} \mu^{n-1} \mathcal{D}(\delta u^{n-1}) : (\mathcal{D}(u^n) + \mathcal{D}(u^{n-1})) \\ \quad + (\Delta d^n + |\nabla d^n|^2 d^n + \Delta d^{n-1} + |\nabla d^{n-1}|^2 d^{n-1}) \\ \quad \times (\Delta \delta d^{n-1} + |\nabla d^n|^2 \delta d^{n-1} + (\nabla d^n + \nabla d^{n-1}) \nabla \delta d^{n-1} d^{n-1}) \\ \operatorname{div} \delta u^n = 0, \quad \int_\Omega \delta P^n dx = 0, \\ (\delta u^n, \delta d^n, \delta \theta^n)|_{t=0} = (0, 0, 0), \quad (\delta u^n, \mathcal{B} \delta d^n, \partial_\nu \delta \theta^n)|_{\partial\Omega} = (0, 0, 0). \end{cases}$$

Applying Theorem 2.1 and Theorem 2.3 to system (3.29), mimicking the process in the second step and noticing that  $\Theta_n(t) + E_n(t) \leq C(\Theta_0 + E_0)$ , for any  $t < T_2$  and  $n \in \mathbb{N}$ , one obtains

$$(3.30) \quad \begin{cases} \delta\Theta_n(t) \leq Ct^{\xi_1}(\delta\Theta_{n-1}(t) + \delta E_{n-1}(t)), \\ \delta E_n(t) \leq Ct^{\xi_2}(\delta\Theta_{n-1}(t) + \delta E_{n-1}(t)), \end{cases}$$

for any  $t < T_2$ ,  $C$  depends on the domain,  $p, q, r, \underline{\mu}, \bar{\mu}, \bar{\mu}', \Theta_0$  and  $E_0$ . Choosing  $T_3 \in (0, T_2]$  such that

$$(3.31) \quad C(T_3^{\xi_1} + T_3^{\xi_2}) \leq \frac{1}{2},$$

then for any  $t < T_3$ , it holds that

$$\delta\Theta_n(t) + \delta E_n(t) \leq \frac{1}{2}(\delta\Theta_{n-1}(t) + \delta E_{n-1}(t)).$$

Therefore,  $(\delta u^n, \delta d^n, \delta P^n, \delta \theta^n)$  is a Cauchy sequence in the Banach space  $E_T^{p,q,r,s}$  for any  $T \leq T_3$ .

**Step 4: Verifying that the limit is a local strong solution.** Let  $(u, d, P, \theta) \in E_T^{p,q,r,s}$  be the limit of  $(u^n, d^n, P^n, \theta^n)_{n \in \mathbb{N}}$  for  $T \leq T_3$ .

We claim that all the nonlinear terms of (3.7) converge to their corresponding terms of (1.1) in  $L_{T_3}^s(L^q)$ . We take one term as an example,

$$\begin{aligned} & \left\| \mu(\theta^n) |\nabla u^n|^2 - \mu(\theta) |\nabla u|^2 \right\|_{L_{T_3}^s(L^q)} \\ & \leq 2\bar{\mu} \left\| \nabla u^n - \nabla u \right\|_{L_{T_3}^{2s}(L^{2q})} \left( \left\| \nabla u \right\|_{L_{T_3}^{2s}(L^{2q})} + \left\| \nabla u^n \right\|_{L_{T_3}^{2s}(L^{2q})} \right) \\ & \quad + \bar{\mu}' \left\| (\theta^n - \theta) |\nabla u^n|^2 \right\|_{L_{T_3}^s(L^q)} \\ & \leq C(E_n(T_3) + E(T_3)) \left( \|u^n - u\|_{E_{T_3}^{p,q,r,s}} + \|\theta^n - \theta\|_{E_{T_3}^{p,q,r,s}} \right) \\ & \rightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

The convergence of the rest terms can be proved in the same spirit as above. Similarly, the nonlinear terms of (3.8) converges in the space  $L_{T_3}^p(W^{1,r})$  and ones of (3.9) converges in the space  $L_{T_3}^p(L^r)$ . Therefore, we can perform the limiting process and it's easy to verify that the limits indeed satisfy system (1.1) with (1.2) almost everywhere.

Next, we check that  $|d| = 1$  *a.e.* on  $\Omega_{T_3}$ . In particular, consider the equations

$$(3.32) \quad \begin{cases} \partial_t d - \Delta d + u \cdot \nabla d = |\nabla d|^2 d, \\ d|_{t=0} = d_0, \quad \text{on } \Omega, \\ d = d_0, \quad \text{or } \partial_\nu d = 0, \quad \text{on } \partial\Omega \times [0, T_3]. \end{cases}$$

Since  $d \in L_{T_3}^\infty(B_{r,p}^{3-\frac{2}{p}}) \cap L_{T_3}^p(W^{3,r})$ , multiplying (3.32) by  $d$  and using the fact that  $\Delta(|d|^2) = 2\Delta d \cdot d + 2|\nabla d|^2$ , one obtains

$$(3.33) \quad \begin{cases} \partial_t(|d|^2 - 1) - \Delta(|d|^2 - 1) + u \cdot \nabla(|d|^2 - 1) = 2|\nabla d|^2(|d|^2 - 1), \\ (|d|^2 - 1)|_{t=0} = 0, \quad \text{on } \Omega, \\ |d|^2 - 1 = 0, \quad \text{or } \partial_\nu(|d|^2 - 1) = 0, \quad \text{on } \partial\Omega \times [0, T_3]. \end{cases}$$

Multiplying (3.33) by  $|d|^2 - 1$ , then integrating the resulting equation over  $\Omega$ , one has

$$(3.34) \quad \frac{1}{2} \frac{d}{dt} \int_{\Omega} (|d|^2 - 1)^2 dx + \int_{\Omega} |\nabla(|d|^2 - 1)|^2 dx = 2 \int_{\Omega} |\nabla d|^2 (|d|^2 - 1)^2 dx.$$

Notice that if  $p \geq 2$ ,  $L_{T_3}^p(W^{3,r}) \hookrightarrow L_{T_3}^2(W^{1,\infty})$ , and if  $1 < p < 2$ ,  $W_{T_3}^{1,p}(W^{1,r}) \cap L_{T_3}^p(W^{3,r}) \hookrightarrow L_{T_3}^2(W^{1,\infty})$ . Thus, by Gronwall's inequality, one has

$$(3.35) \quad \int_{\Omega} (|d|^2 - 1)^2(t, x) dx \leq \exp \left( C \int_0^t \|\nabla d(s, \cdot)\|_{L^\infty}^2 ds \right) \int_{\Omega} (|d|^2 - 1)^2(0, x) dx = 0,$$

for any  $t < T_3$ . This implies that  $|d| = 1$  *a.e.* on  $\Omega_{T_3}$ .

Finally, the existence of local strong solution is proved. The proof of the uniqueness and continuity is standard, here we omit the details.

#### 4. GLOBAL EXISTENCE FOR SMALL PERTURBATION

It is showed in the last section in implicit form that if the initial data  $\Theta_0$  and  $E_0$  are smaller, the lifespan of local strong solutions is longer. In this section we are going to prove that actually for sufficiently small perturbation around the trivial equilibrium state  $(0, \mathbf{e}, 0)$ , the strong solution is global.

*Proof of Theorem 1.3.* Suppose  $T^*$  is the maximal existence time and fix  $t < T^*$ . Define

$$\begin{aligned} U(t) &:= \|u\|_{L_t^\infty(D_{A_r}^{1-\frac{1}{p}, p})} + \|u\|_{W_t^{1,p}(L^r) \cap L_t^p(W^{2,r})} + \|P\|_{L_t^p(W^{1,r})}, \\ D(t) &:= \|\bar{d}\|_{L_t^\infty(B_{r,p}^{3-\frac{2}{p}})} + \|\bar{d}\|_{W_t^{1,p}(W^{1,r}) \cap L_t^p(W^{3,r})}, \\ \Theta(t) &:= \|\theta\|_{L_t^\infty(B_{q,s}^{2-\frac{2}{s}})} + \|\theta\|_{W_t^{1,s}(L^q) \cap L_t^s(W^{2,q})}, \\ F(t) &:= U(t) + D(t) + \Theta(t), \quad F_0 := U_0 + D_0 + \Theta_0, \end{aligned}$$

where  $U_0$ ,  $D_0$  and  $\Theta_0$  are defined as before,  $\bar{d} := d - \mathbf{e}$ .

Applying Theorem 2.1 to the temperature equation in system (1.1), one has

$$\begin{aligned} (4.1) \quad \Theta(t) &\leq C(\Theta_0 + \|u \cdot \nabla \theta\|_{L_t^s(L^q)} + \bar{\mu} \|\nabla u\|^2_{L_t^q(L^q)} \\ &\quad + \|\Delta \bar{d} + |\nabla \bar{d}|^2(\bar{d} + \mathbf{e})\|_{L_t^s(L^q)}) \\ &\leq C(\Theta_0 + U(t)\Theta(t) + U^2(t) + D^2(t) + D^6(t)), \end{aligned}$$

where we have used Lemma 3.3. Then applying Theorem 2.1 to the second equation of (1.1), it follows that

$$\begin{aligned} (4.2) \quad D(t) &\leq C(D_0 + \|u \cdot \nabla \bar{d}\|_{L_t^p(W^{1,r})} + \|\nabla \bar{d}\|^2(\bar{d} + \mathbf{e})\|_{L_t^p(W^{1,r})}) \\ &\leq C(D_0 + \|u\|_{L_t^\infty(L^r)} \|\nabla \bar{d}\|_{L_t^p(W^{1,\infty})} + \|\nabla u\|_{L_t^p(L^\infty)} \|\nabla \bar{d}\|_{L_t^\infty(L^r)} \\ &\quad + \|\nabla \bar{d}\|_{L_t^p(W^{1,\infty})} \|\nabla \bar{d}\|_{L_t^\infty(L^r)} (1 + \|\bar{d}\|_{L_t^\infty(W^{1,\infty})})) \\ &\leq C(D_0 + U(t)D(t) + D^2(t) + D^3(t)), \end{aligned}$$

where we have used Lemma 3.3 and  $B_{r,p}^{3-\frac{2}{p}} \hookrightarrow W^{1,\infty}$ . Applying Theorem 2.3 to the velocity equation in system (1.1) and using (2.7), one reaches

$$(4.3) \quad \begin{aligned} U(t) \leq & C(1 + \Theta(t))^{\frac{kq}{q-N}} (U_0 + \|u \cdot \nabla u\|_{L_t^p(L^r)} + \|\Delta \bar{d} \cdot \nabla \bar{d}\|_{L_t^p(L^r)}) \\ & + (1 + \Theta(t))^{\frac{q_1}{q-N}} (\|\nabla \theta\|_{L_t^\infty(L^q)} + \|\theta\|_{\dot{C}_t^\beta(L^\infty)})^{l_2} \|u\|_{L_t^p(L^r)}. \end{aligned}$$

Noticing that  $W_t^{1,s}(L^q) \cap L_t^s(W^{2,q}) \hookrightarrow \dot{C}_t^\beta(L^\infty)$ , it follows that

$$(4.4) \quad U(t) \leq C(1 + \Theta(t))^{\max\{k,l_1\}\frac{q}{q-N}} (U_0 + U^2(t) + D^2(t) + \Theta^{l_2}(t)U(t)).$$

Summing up (4.1), (4.2) and (4.4), it yields that

$$(4.5) \quad F(t) \leq C(1 + F(t))^{\max\{k,l_1\}\frac{q}{q-N}} (F_0 + F^2(t) + F^3(t) + F^6(t) + F^{1+l_2}(t)).$$

Assume that for some fixed  $T > 0$  such that

$$(4.6) \quad F(t) \leq 5CF_0, \quad \forall t \in [0, T].$$

If the initial data is sufficiently small such that

$$(4.7) \quad \begin{cases} (1 + 5CF_0)^{\max\{k,l_1\}\frac{q}{q-N}} \leq 2, \\ 5^2 C^2 F_0 + 5^3 C^3 F_0^2 + M^6 C^6 F_0^5 + 5^{1+l_2} C^{1+l_2} F_0^{l_2} \leq 1, \end{cases}$$

then one can be convinced that, by (4.5),

$$(4.8) \quad F(t) \leq 4CF_0, \quad \forall t \in [0, T].$$

Thus by a continuation argument, one can extend a local solution to a global one.  $\delta$  in the smallness condition of initial data is determined by (4.7).  $\square$

## 5. APPENDIX: PROOF OF THEOREM 2.3

First of all, recall the maximal regularity for the linear Stokes operator (cf. Theorem 3.2 [8]):

**Theorem 5.1.** *Let  $\Omega \subset \mathbb{R}^N$  be a bounded domain with smooth boundary and  $1 < p, r < \infty$ .  $u_0 \in D_{A_r}^{1-\frac{1}{p},p}$ ,  $f \in L^p(0, T; L^r)$  and  $\mu$  is a positive constant. Then the system*

$$\begin{cases} \partial_t u - \mu \Delta u + \nabla P = f, \\ \operatorname{div} u = 0, \quad \int_\Omega P \, dx = 0, \\ u|_{t=0} = u_0, \quad u|_{\partial\Omega} = 0, \end{cases}$$

has a unique global solution  $(u, P)$  satisfying

$$\begin{aligned} & \mu^{1-\frac{1}{p}} \|u\|_{L^\infty(0, T; D_{A_r}^{1-\frac{1}{p},p})} + \|(\partial_t u, \mu \Delta u, u, \nabla P)\|_{L^p(0, T; L^r)} \\ & \leq C \left( \mu^{1-\frac{1}{p}} \|u_0\|_{D_{A_r}^{1-\frac{1}{p},p}} + \|f\|_{L^p(0, T; L^r)} \right), \end{aligned}$$

for all  $T \geq 0$ , with  $C = C(p, r, \sigma(\Omega))$ , where  $\sigma(\Omega)$  stands for the open set

$$\sigma(\Omega) = \left\{ \frac{x}{\delta(\Omega)} \mid x \in \Omega \right\},$$

with  $\delta(\Omega)$  denoting the diameter of  $\Omega$ .

*Remark 5.2.* Notice that in the above estimates  $C$  depends on the shape of the domain  $\Omega$ , but is independent of the diameter  $\delta(\Omega)$ .

The basic idea is that if  $\theta$  is close to a constant  $\underline{\theta}$ , Theorem 5.1 provides us with the desired estimates. Indeed, one can rewrite the system as

$$(5.1) \quad \begin{cases} \partial_t u - \operatorname{div}(\mu(\underline{\theta})\mathcal{D}(u)) + \nabla P = f + (\mu(\theta) - \mu(\underline{\theta}))\Delta u + \mu'(\theta)\nabla\theta \cdot \mathcal{D}(u), \\ \operatorname{div} u = 0, \quad \int_{\Omega} P \, dx = 0, \\ u|_{t=0} = u_0, \quad u|_{\partial\Omega} = 0. \end{cases}$$

Now if  $\|\mu(\theta) - \mu(\underline{\theta})\|_{L^\infty}$  is small, the term  $\|(\mu(\theta) - \mu(\underline{\theta}))\Delta u\|_{L_t^p(L^r)}$  may be absorbed by the left-hand side (LHS) of the inequality given in Theorem 5.1. Although generally one can not expect  $\theta$  is close enough to a constant, but if  $\theta$  is Lipschitz continuous, it will not deviate from a constant too much in small enough domain. Thus one can perform localization argument to recover Stokes estimates.

The proof of Theorem 2.3 is organized as follows. First, we restrict ourselves to the case of null initial data, *i.e.*,  $u_0 \equiv 0$ , and prove the *a priori* estimates for  $(u, P)$  under the assumption that  $\theta$  is independent of time. Next, we prove the similar estimates for time-dependent temperature. Finally, we derive the desired estimates for the general initial data  $u_0 \in D_{A_r}^{1-\frac{1}{p}, p}$ .

5.0.1. *Existence of solution for null initial data.* We divide the proof into the following three steps.

(a) *A priori estimates with time-independent temperature*

**Theorem 5.3.** *Suppose  $p, q, r, \mu$  and  $\Omega$  satisfy the assumptions in Theorem 2.3,  $u_0 = 0$  and  $\theta = \theta(x) \in W^{1,q}$ . If  $(u, P)$  is a smooth solution to system (2.5) on  $\Omega \times [0, T)$ , then for any  $t < T$  it holds that*

$$(5.2) \quad \begin{aligned} & \|(\partial_t u, \Delta u, \nabla P)\|_{L_t^p(L^r)} \\ & \leq C \left( B_\theta^{1+\tilde{\zeta}} \|f\|_{L_t^p(L^r)} + (B_\theta - 1)^{2r'(\tilde{\zeta} \cdot \frac{q}{q-N} + 1)} \|u\|_{L_t^p(L^r)} \right), \end{aligned}$$

where  $B_\theta = 1 + \|\nabla\theta\|_{L^q(\Omega)}^{\frac{q}{q-N}}$ ,  $\tilde{\zeta} = \max\{0, \frac{N}{p} - \frac{N}{r}\}$  and  $C$  is independent of  $f, \theta$  and  $t$ .

*Proof.* Rewriting (2.5) as (5.1) and applying Theorem 5.1, we obtain

$$(5.3) \quad \begin{aligned} & \|u\|_{L_T^\infty(D_{A_r}^{1-\frac{1}{p}, p})} + \|(\partial_t u, \Delta u, \nabla P)\|_{L_T^p(L^r)} \\ & \leq C (\|f\|_{L_T^p(L^r)} + \|\mu(\theta) - \mu(\underline{\theta})\|_{L^\infty} \|\Delta u\|_{L_T^p(L^r)} + \|\nabla u \nabla \theta\|_{L_T^p(L^r)}), \end{aligned}$$

where  $\underline{\theta} = \inf_{x \in \Omega} \theta(x)$  and  $C$  depends on  $p, q, r, \Omega, \bar{\mu}, \underline{\mu}$  and  $\bar{\mu}'$ . From now on, we will keep this dependence of  $C$  in silence unless otherwise claimed, the value of  $C$  may change from line to line.

By Gagliardo-Nirenberg interpolation inequality, Poincaré-Wirtinger inequality and Young's inequality, we arrive at

$$(5.4) \quad \|\nabla u \nabla \theta\|_{L_T^p(L^r(\Omega))} \leq \epsilon \|\Delta u\|_{L_T^p(L^r(\Omega))} + \epsilon^{\frac{N+q}{N-q}} \|\nabla\theta\|_{L^q(\Omega)}^{\frac{2q}{q-N}} \|u\|_{L_T^p(L^r(\Omega))},$$

for any  $\epsilon > 0$ .

On the other hand, since  $q > N$ ,  $\theta \in W^{1,q}(\Omega) \hookrightarrow \dot{C}^\alpha(\Omega)$  for  $\alpha = 1 - \frac{N}{q} \in (0, 1)$ , we have

$$(5.5) \quad \|\mu(\theta) - \mu(\underline{\theta})\|_{L^\infty} \leq \bar{\mu}' \|\theta - \underline{\theta}\|_{L^\infty(\Omega)} \leq \bar{\mu}' \delta^\alpha(\Omega) \|\theta\|_{\dot{C}^\alpha(\Omega)} \leq \bar{\mu}' \delta^\alpha(\Omega) \|\nabla\theta\|_{L^q(\Omega)}.$$

If  $C\delta^\alpha(\Omega)\|\nabla\theta\|_{L^q(\Omega)} \leq \frac{1}{4}$ , then the corresponding term is absorbed by the LHS of (5.3). Choosing  $\epsilon = \frac{1}{4C}$  and substituting (5.4) and (5.5) into (5.3), we obtain

$$(5.6) \quad \begin{aligned} & \|u\|_{L_T^\infty(D_{A_r}^{\frac{1}{p},p})} + \|(\partial_t u, \Delta u, \nabla P)\|_{L_T^p(L^r(\Omega))} \\ & \leq C(\|f\|_{L_T^p(L^r(\Omega))} + \|\nabla\theta\|_{L^q(\Omega)}^{\frac{2q}{q-N}} \|u\|_{L_T^p(L^r(\Omega))}). \end{aligned}$$

Otherwise if  $C\delta^\alpha(\Omega)\|\nabla\theta\|_{L^q(\Omega)} > \frac{1}{4}$ , we perform the space localization to adjust  $\delta(\Omega)$ .

We consider the following subordinate partition of  $\Omega$ :

$\{\Omega_k\}_{k=1}^K$  is an open covering of  $\Omega$  with multiplicity  $m$ ,<sup>1</sup>  $\Omega_k$  is star-shaped and for  $1 \leq k \leq K$ , it holds that

$$\delta(\Omega_k) \leq \lambda \in (0, \delta(\Omega)),$$

the value of  $\lambda$  will be determined later.  $\{\phi_k\}_{k=1}^K$  is a family of characteristic function such that

$$\begin{aligned} 0 \leq \phi_k \leq 1, \quad \phi_k \in C_c^2(\Omega_k), \quad \sum_{k=1}^K \phi_k(x) = 1, \quad \forall x \in \bar{\Omega}, \\ \|\nabla^\alpha \phi_k\|_{L^\infty(\Omega_k)} \leq C\lambda^{-|\alpha|}, \quad |\alpha| \leq 2. \end{aligned}$$

The number  $K$  of the covering is of order  $(\delta(\Omega)\lambda^{-1})^N$ ,

and the number  $K'$  of domains  $\Omega_k$  intersecting with  $\partial\Omega$  is of order  $(\delta(\Omega)\lambda^{-1})^{N-1}$ .

Now define  $u_k = u\phi_k$ ,  $P_k = P\phi_k$  and  $f_k = f\phi_k$ . Then  $(u_k, P_k, f_k)$  satisfies the following system

$$(5.7) \quad \begin{cases} \partial_t u_k - \mu(\underline{\theta}_k)\Delta u_k + \nabla P_k = f_k - 2\mu(\theta)\nabla u \cdot \nabla \phi_k + \mu'(\theta)\mathcal{D}(u) \cdot \nabla \theta \phi_k \\ \quad - \mu(\theta)u\Delta \phi_k + P\nabla \phi_k + (\mu(\theta) - \mu(\underline{\theta}_k))\Delta u_k, \\ \operatorname{div} u_k = u \cdot \nabla \phi_k, \quad \int_{\Omega} P_k dx = 0, \\ u_k|_{t=0} = 0, \quad u_k|_{\partial\Omega} = 0, \end{cases}$$

where  $\underline{\theta}_k = \inf_{x \in \Omega_k} \theta(x)$ . Notice that  $u_k$  is not divergence-free and the localization procedure produces some additional lower order terms. To obtain the estimates of the above system, we use a theorem proved by Danchin:

**Theorem 5.4** (Theorem 3.6 in [8]). *Let  $\Omega$  be a  $C^{2+\epsilon}$  bounded domain of  $\mathbb{R}^N$  and  $1 < p, r < \infty$ . Let  $\Omega' \subset \bar{\Omega}$  be open and star-shaped with respect to small ball of diameter  $d > 0$ . Let  $\tau \in L^p(0, T; W^{1,r})$  satisfy  $\tau(0, \cdot) \equiv 0$ ,*

$$\int_{\Omega} \tau dx = 0, \quad \partial_t \tau = \tau_0 + \operatorname{div} R, \quad \text{and} \quad \forall t \in (0, T), \quad \operatorname{supp} \tau_0(t, \cdot) \cap \operatorname{supp} R(t, \cdot) \subset \bar{\Omega}',$$

*with  $R$  and  $\tau_0$  in  $L^p(0, T; L^r(\Omega))$  and  $R \cdot n$  in  $L^p(0, T; L^r(\partial\Omega))$ . Let  $v_0 \in D_{A_r}^{1-\frac{1}{p}, p}$ ,  $f \in L^p(0, T; L^r(\Omega))$  and  $\mu$  is a constant. Then the following system*

$$\begin{cases} \partial_t v - \mu\Delta v + \nabla P = f, \\ \operatorname{div} v = \tau, \quad \int_{\Omega} P dx = 0, \\ v|_{t=0} = v_0, \quad v|_{\partial\Omega} = 0, \end{cases}$$

<sup>1</sup>The multiplicity of an covering means at most how many subsets intersect with each other, this quantity only depends of the space dimension  $N$ .



has a unique solution  $(v, P)$  on  $\Omega \times [0, T)$  such that

$$v \in L^p(0, T; W^{2,r}) \cap W^{1,p}(0, T; L^r) \quad \text{and} \quad P \in L^p(0, T; W^{1,r}).$$

Besides, the following estimate holds true with  $C = C(r, p, N, \sigma(\Omega))$ :

$$\begin{aligned} \|(\partial_t v, \mu \nabla^2 v, \nabla P)\|_{L_T^p(L^r(\Omega))} &\leq C(\mu^{1-\frac{1}{p}} \|v_0\|_{D_{A_r}^{1-\frac{1}{p}, p}} + \|f\|_{L_T^p(L^r(\Omega))} + \|R\|_{L_T^p(L^r(\Omega))} \\ &\quad + \mu \|\nabla \tau\|_{L_T^p(L^r(\Omega))} + \delta(\Omega') \|\tau_0\|_{L_T^p(L^r(\Omega))} \\ &\quad + \delta^{\frac{1}{r}}(\Omega') \|R \cdot n\|_{L_T^p(L^r(\partial\Omega))}). \end{aligned}$$

Let  $\tau = u \cdot \nabla \phi_k$ , then  $\tau(0, \cdot) = u(0, \cdot) \nabla \phi_k \equiv 0$ ,  $\int_{\Omega} \tau \, dx = \int_{\Omega} \operatorname{div} u_k \, dx = 0$ . Moreover,

$$\begin{aligned} \partial_t \tau &= \partial_t u \cdot \nabla \phi_k \\ &= \mu(\theta) \Delta u \cdot \nabla \phi_k + \mu'(\theta) \mathcal{D}(u) \nabla \theta \nabla \phi_k - \nabla P \cdot \nabla \phi_k + f \nabla \phi_k \\ &= \underbrace{f \nabla \phi_k + P \Delta \phi_k - \mu(\theta) \mathcal{D}(u) \Delta \phi_k}_{\tau_0} + \underbrace{\operatorname{div}(\mu(\theta) \mathcal{D}(u) \cdot \nabla \phi_k - P \nabla \phi_k)}_R, \end{aligned}$$

and  $\operatorname{supp} \tau_0(t, \cdot) \cap \operatorname{supp} R(t, \cdot) \subset \bar{\Omega}_k$ . Hence by Theorem 5.4, there exists a unique solution  $(u_k, P_k)$  to (5.7) satisfying

$$\begin{aligned} (5.8) \quad &\|(\partial_t u_k, \Delta u_k, \nabla P_k)\|_{L^p(0, T; L^r(\Omega))} \\ &\lesssim \|g_k\|_{L_T^p(L^r(\Omega))} + \|R\|_{L_T^p(L^r(\Omega))} + \lambda \|\tau_0\|_{L_T^p(L^r(\Omega))} + \lambda^{\frac{1}{r}} \|R \cdot n\|_{L_T^p(L^r(\partial\Omega))} \\ &\quad + \|\nabla \tau\|_{L_T^p(L^r(\Omega))} + \|\mu(\theta) - \mu(\underline{\theta}_k)\|_{L^\infty(\Omega_k)} \|\Delta u_k\|_{L_T^p(L^r(\Omega))}, \end{aligned}$$

where  $g_k = f_k + \mu'(\theta) \mathcal{D}(u) \nabla \theta \nabla \phi_k - 2\mu(\theta) \nabla u \cdot \nabla \phi_k - \mu(\theta) u \Delta \phi_k + P \nabla \phi_k$ .

First of all,

$$(5.9) \quad \|\mu(\theta) - \mu(\underline{\theta}_k)\|_{L^\infty(\Omega_k)} \|\Delta u_k\|_{L_T^p(L^r(\Omega))} \leq \bar{\mu}' \lambda^\alpha \|\nabla \theta\|_{L^q(\Omega)} \|\Delta u_k\|_{L_T^p(L^r(\Omega))}.$$

Choose  $\lambda \leq \kappa \|\theta\|_{W^{1,q}(\Omega)}^{-\frac{1}{\alpha}}$  with  $\kappa \ll 1$  such that the corresponding term can be absorbed by the LHS of (5.8). Next, we move on to evaluate the terms on the RHS of (5.8) one by one.

$$\begin{aligned} (5.10) \quad &\|g_k\|_{L_T^p(L^r(\Omega))} \lesssim \|f_k\|_{L_T^p(L^r(\Omega))} + \|\nabla u \cdot \nabla \phi_k\|_{L_T^p(L^r(\Omega))} \\ &\quad + \|\nabla u \nabla \theta \nabla \phi_k\|_{L_T^p(L^r(\Omega))} + \|u \Delta \phi_k\|_{L_T^p(L^r(\Omega))} \\ &\quad + \|P \nabla \phi_k\|_{L_T^p(L^r(\Omega))} \\ &\lesssim \|f\|_{L_T^p(L^r(\Omega_k))} + \lambda^{-1} \|\nabla u\|_{L_T^p(L^r(\Omega_k))} \\ &\quad + \|\nabla u \nabla \theta\|_{L_T^p(L^r(\Omega_k))} + \lambda^{-2} \|u\|_{L_T^p(L^r(\Omega_k))} \\ &\quad + \lambda^{-1} \|P\|_{L_T^p(L^r(\Omega_k))}. \end{aligned}$$

Similarly, one also has

$$(5.11) \quad \|R\|_{L_T^p(L^r(\Omega))} \lesssim \lambda^{-1} \|\nabla u\|_{L_T^p(L^r(\Omega_k))} + \lambda^{-1} \|P\|_{L_T^p(L^r(\Omega_k))},$$

$$(5.12) \quad \lambda \|\tau_0\|_{L_T^p(L^r(\Omega))} \lesssim \|f\|_{L_T^p(L^r(\Omega_k))} + \lambda^{-1} \|P\|_{L_T^p(L^r(\Omega_k))} + \lambda^{-1} \|\nabla u\|_{L_T^p(L^r(\Omega_k))},$$

$$(5.13) \quad \lambda^{\frac{1}{r}} \|R \cdot n\|_{L_T^p(L^r(\partial\Omega))} \lesssim \lambda^{-1+\frac{1}{r}} (\|\nabla u\|_{L_T^p(L^r(\partial\Omega \cap \Omega_k))} + \|P\|_{L_T^p(L^r(\partial\Omega \cap \Omega_k))}).$$

Substituting (5.9)-(5.13) into (5.8), one reaches

$$\begin{aligned}
(5.14) \quad & \|(\partial_t u_k, \Delta u_k, \nabla P_k)\|_{L_T^p(L^r(\Omega))} \\
& \lesssim \|f\|_{L_T^p(L^r(\Omega_k))} + \lambda^{-1} \|\nabla u\|_{L_T^p(L^r(\Omega_k))} + \lambda^{-2} \|u\|_{L_T^p(L^r(\Omega_k))} \\
& \quad + \lambda^{-1} \|P\|_{L_T^p(L^r(\Omega_k))} + \|\nabla u \nabla \theta\|_{L_T^p(L^r(\Omega_k))} \\
& \quad + \lambda^{-\frac{1}{r'}} (\|\nabla u\|_{L_T^p(L^r(\partial\Omega \cap \Omega_k))} + \|P\|_{L_T^p(L^r(\partial\Omega \cap \Omega_k))}).
\end{aligned}$$

Next, we sum up the local estimates to obtain the whole estimates. Noticing that

$$\forall z \in L_T^p(L^r(\Omega)), \quad \sum_{k=1}^K \|z\|_{L_T^p(L^r(\Omega_k))} \leq m^{\frac{1}{r}} K^\zeta \|z\|_{L_T^p(L^r(\Omega))},$$

where  $\zeta = \max\{0, \frac{1}{p} - \frac{1}{r}\}$ , summing up (5.14) over  $k$ , we have

$$\begin{aligned}
(5.15) \quad & \|(\partial_t u, \Delta u, \nabla P)\|_{L_T^p(L^r(\Omega))} \\
& \lesssim \lambda^{-N\zeta} (\|f\|_{L_T^p(L^r(\Omega))} + \lambda^{-1} \|\nabla u\|_{L_T^p(L^r(\Omega))} \\
& \quad + \lambda^{-2} \|u\|_{L_T^p(L^r(\Omega))} + \lambda^{-1} \|P\|_{L_T^p(L^r(\Omega))} + \|\nabla u \nabla \theta\|_{L_T^p(L^r(\Omega))}) \\
& \quad + \lambda^{-(N-1)\zeta} \lambda^{-\frac{1}{r'}} (\|\nabla u\|_{L_T^p(L^r(\partial\Omega))} + \|P\|_{L_T^p(L^r(\partial\Omega))}).
\end{aligned}$$

Standard interpolation inequalities enable us to further simplify the RHS of (5.15). By Gagliardo-Nirenberg and Young's inequality, it follows that

$$(5.16) \quad \|\nabla u\|_{L^r(\Omega)} \leq C(\eta_1^{-1} \|u\|_{L^r(\Omega)} + \eta_1 \|\nabla^2 u\|_{L^r(\Omega)}), \quad \forall \eta_1 > 0.$$

And according to the trace theorem (page 63 in [14]), one deduces that

$$(5.17) \quad \|P\|_{L^r(\partial\Omega)} \leq C(\eta_2^{-\frac{1}{r}} \|P\|_{L^r(\Omega)} + \eta_2^{\frac{1}{r'}} \|\nabla P\|_{L^r(\Omega)}), \quad \forall \eta_2 > 0,$$

$$(5.18) \quad \|\nabla u\|_{L^r(\partial\Omega)} \leq C(\eta_3^{-\frac{1}{r}-1} \|u\|_{L^r(\Omega)} + \eta_3^{\frac{1}{r'}} \|\nabla^2 u\|_{L^r(\Omega)}), \quad \forall \eta_3 > 0.$$

Again by (5.4),

$$(5.19) \quad \|\nabla u \nabla \theta\|_{L^r(\Omega)} \leq C(\eta_4 \|\nabla^2 u\|_{L^r(\Omega)} + \eta_4^{\frac{N+q}{N-q}} \|\nabla \theta\|_{L^q(\Omega)}^{\frac{2q}{q-N}} \|u\|_{L^r(\Omega)}), \quad \forall \eta_4 > 0.$$

Now choose  $\eta_1 = \epsilon \lambda^{N\zeta+1}$ ,  $\eta_2 = \eta_3 = \epsilon \lambda^{(N-1)\zeta r'+1}$  and  $\eta_4 = \epsilon \lambda^{N\zeta}$ , with  $\epsilon \ll 1$ , then the terms  $\|\nabla P\|_{L_T^p(L^r(\Omega))}$  and  $\|\nabla^2 u\|_{L_T^p(L^r(\Omega))}$  can be absorbed by the LHS of (5.15). Consequently, substituting (5.16)-(5.19) into (5.15), we reach

$$\begin{aligned}
(5.20) \quad & \|(\partial_t u, \Delta u, \nabla P)\|_{L_T^p(L^r(\Omega))} \\
& \lesssim \lambda^{-N\zeta} \|f\|_{L_T^p(L^r(\Omega))} + \lambda^{-2-N\zeta \cdot \frac{2q}{q-N}} \|u\|_{L_T^p(L^r(\Omega))} + \lambda^{-1-N\zeta} \|P\|_{L_T^p(L^r(\Omega))},
\end{aligned}$$

where we have used  $r > N$ .

It remains to show the pressure estimates in terms of  $f$  and  $u$  to complete the proof. To this end, we evaluate  $P$  by a duality argument

$$\|P\|_{L^r(\Omega)} = \sup_{\substack{\|h\|_{L^{r'}(\Omega)} \leq 1, \\ \int_{\Omega} h \, dx = 0}} \int_{\Omega} P h \, dx.$$

Let

$$(5.21) \quad \Delta v = h, \quad \partial_\nu v|_{\partial\Omega} = 0.$$

Then according to Proposition C.1 in [8], we have

$$(5.22) \quad \|\nabla v\|_{L^{r'}(\Omega)} \leq C(\Omega)\|h\|_{L^{r'}(\Omega)}, \quad \|\nabla^2 v\|_{L^{r'}(\Omega)} \leq C(\Omega)\|h\|_{L^{r'}(\Omega)}.$$

Hence,

$$(5.23) \quad \begin{aligned} \int_{\Omega} Ph \, dx &= \int_{\Omega} P \Delta v \, dx = - \int_{\Omega} \nabla P \cdot \nabla v \, dx \\ &= - \int_{\Omega} \left( f - \partial_t u + \operatorname{div}(\mu(\theta) \nabla u) \right) \cdot \nabla v \, dx \\ &\lesssim \|f\|_{L^r(\Omega)} \|\nabla v\|_{L^{r'}(\Omega)} + \|\nabla u\|_{L^r(\partial\Omega)} \|\nabla v\|_{L^{r'}(\partial\Omega)} \\ &\quad + \|\nabla u\|_{L^r(\Omega)} \|\Delta v\|_{L^{r'}(\Omega)} \\ &\lesssim (\|f\|_{L^r(\Omega)} + \|\nabla u\|_{L^r(\Omega)}) \|h\|_{L^{r'}(\Omega)} + \|\nabla u\|_{L^r(\partial\Omega)} \|\nabla v\|_{L^{r'}(\partial\Omega)}, \end{aligned}$$

where we have used (5.22).

For  $\nabla u$ , by interpolation and Young's inequality, it holds that

$$(5.24) \quad \|\nabla u\|_{L^r(\Omega)} \lesssim \epsilon_1^{-1} \|u\|_{L^r(\Omega)} + \epsilon_1 \|\nabla^2 u\|_{L^r(\Omega)}, \quad \forall \epsilon_1 > 0.$$

Finally, one can use trace theorem to simplify the boundary terms as

$$(5.25) \quad \|\nabla v\|_{L^{r'}(\partial\Omega)} \lesssim \|\nabla v\|_{L^r(\Omega)}^{\frac{1}{r}} (\|\nabla^2 v\|_{L^{r'}(\Omega)} + \|\nabla v\|_{L^{r'}(\Omega)})^{\frac{1}{r'}} \lesssim \|h\|_{L^{r'}(\Omega)},$$

$$(5.26) \quad \|\nabla u\|_{L^r(\partial\Omega)} \lesssim \epsilon_2^{-1-\frac{1}{r}} \|u\|_{L^r(\Omega)} + \epsilon_2^{\frac{1}{r'}} \|\nabla^2 u\|_{L^r(\Omega)}, \quad \forall \epsilon_2 > 0.$$

Substituting (5.24)-(5.26) into (5.23), one obtains

$$(5.27) \quad \|P\|_{L^r(\Omega)} \lesssim \|f\|_{L^r(\Omega)} + (\epsilon_1^{-1} + \epsilon_2^{-1-\frac{1}{r}}) \|u\|_{L^r(\Omega)} + (\epsilon_1 + \epsilon_2^{\frac{1}{r'}}) \|\nabla^2 u\|_{L^r(\Omega)}.$$

Plugging (5.27) into (5.20), and choosing

$$\epsilon_1 = \kappa \lambda^{1+N\zeta}, \quad \epsilon_2 = \kappa \lambda^{r'+r'N\zeta},$$

where  $\kappa \ll 1$ , then  $\|\nabla^2 u\|_{L_T^p(L^r(\Omega))}$  can be absorbed by the LHS of (5.20), which implies that

$$(5.28) \quad \begin{aligned} \|(\partial_t u, \Delta u, \nabla P)\|_{L_T^p(L^r(\Omega))} &\lesssim \lambda^{-N\zeta-1} \|f\|_{L_T^p(L^r(\Omega))} \\ &\quad + \lambda^{-2r'(1+N\zeta\frac{q}{q-N})} \|u\|_{L_T^p(L^r(\Omega))}. \end{aligned}$$

Combining (5.6) with (5.28), one finally obtains (5.2). The proof of Theorem 5.3 is completed.  $\square$

(b) *A priori estimates with time-dependent temperature*

Based on Theorem 5.3, we generalize the above results to the case of time-dependent temperature. The main result is the following.

**Theorem 5.5.** *Suppose  $p, q, r, \Omega$  and  $\mu$  satisfy the assumptions in Theorem 2.3,  $u_0 = 0$  and the temperature satisfies*

$$(5.29) \quad \theta \in \dot{C}^\beta([0, T]; L^\infty(\Omega)) \cap L^\infty(0, T; W^{1,q}(\Omega)),$$

for some  $\beta \in (0, 1)$ . If  $(u, P)$  is a smooth solution to (2.5) on  $\Omega \times [0, T)$ , then it holds that

$$(5.30) \quad \|(\partial_t u, \Delta u, \nabla P)\|_{L_T^p(L^r(\Omega))} \leq C(B_\theta^{1+\zeta}(T) \|f\|_{L_t^p(L^r(\Omega))} + C_\theta(T) \|u\|_{L_t^p(L^r(\Omega))}),$$

where  $B_\theta(t)$ ,  $C_\theta(t)$  and  $C$  are defined as in Theorem 2.3.

*Proof.* First, rewrite (2.5) as the following system

$$(5.31) \quad \begin{cases} \partial_t u - \operatorname{div}(\mu(\theta_0)\mathcal{D}(u)) + \nabla P = f + (\mu(\theta) - \mu(\theta_0))\Delta u \\ \quad + (\mu'(\theta)\nabla\theta - \mu(\theta_0)\nabla\theta_0)\mathcal{D}(u), \\ \operatorname{div} u = 0, \quad \int_{\Omega} P \, dx = 0, \\ u|_{t=0} = 0, \quad u|_{\partial\Omega} = 0. \end{cases}$$

Applying Theorem 5.3 to system (5.31), we have for any  $t < T$

$$(5.32) \quad \begin{aligned} & \|(\partial_t u, \Delta u, \nabla P)\|_{L_t^p(L^r(\Omega))} \\ & \lesssim B_{\theta_0}^{1+\tilde{\zeta}} (\|f\|_{L_t^p(L^r(\Omega))} + \|(\mu(\theta) - \mu(\theta_0))\Delta u\|_{L_t^p(L^r(\Omega))} \\ & \quad + \|(\mu'(\theta)\nabla\theta - \mu'(\theta_0)\nabla\theta_0)\nabla u\|_{L_t^p(L^r(\Omega))}) \\ & \quad + (B_{\theta_0} - 1)^{2r'(1+\tilde{\zeta})\cdot\frac{q}{q-N}} \|u\|_{L_t^p(L^r(\Omega))}. \end{aligned}$$

Notice that

$$\begin{aligned} & \|(\mu'(\theta)\nabla\theta - \mu'(\theta_0)\nabla\theta_0)\nabla u\|_{L_t^p(L^r(\Omega))} \\ & \lesssim \epsilon \|\nabla^2 u\|_{L_t^p(L^r(\Omega))} + \epsilon^{\frac{N+q}{N-q}} (\|\nabla\theta\|_{L_t^\infty(L^q(\Omega))} + \|\nabla\theta_0\|_{L^q(\Omega)})^{\frac{2q}{q-N}} \|u\|_{L_t^p(L^r(\Omega))}. \end{aligned}$$

Thus choose  $\epsilon = \kappa B_{\theta_0}^{-1-\tilde{\zeta}}$  with  $\kappa \ll 1$  such that  $\nabla^2 u$  can be absorbed by the LHS of (5.32). On the other hand,

$$\begin{aligned} \|(\mu(\theta) - \mu(\theta_0))\Delta u\|_{L_t^p(L^r(\Omega))} & \leq \bar{\mu}' \|\theta - \theta_0\|_{L^\infty(\Omega_t)} \|\Delta u\|_{L_t^p(L^r(\Omega))} \\ & \leq \bar{\mu}' t^\beta \|\theta\|_{\dot{C}_t^\beta(L^\infty(\Omega))} \|\Delta u\|_{L_t^p(L^r(\Omega))}. \end{aligned}$$

Substituting the above two inequalities into (5.32), we obtain

$$(5.33) \quad \begin{aligned} & \|(\partial_t u, \Delta u, \nabla P)\|_{L_t^p(L^r(\Omega))} \\ & \lesssim B_{\theta_0}^{1+\tilde{\zeta}} (\|f\|_{L_t^p(L^r(\Omega))} + t^\beta \|\theta\|_{\dot{C}_t^\beta(L^\infty)} \|\Delta u\|_{L_t^p(L^r(\Omega))}) \\ & \quad + B_\theta^{(1+\tilde{\zeta})\cdot\frac{2q}{q-N}}(t) (B_\theta(t) - 1)^{(1+\tilde{\zeta})\cdot\frac{2qr'}{q-N}+2} \|u\|_{L_t^p(L^r(\Omega))}, \end{aligned}$$

where  $B_\theta(t) = 1 + \|\nabla\theta\|_{\dot{C}_t^\beta(L^\infty(\Omega))}^{\frac{q}{q-N}}$ .

If  $t^\beta \|\theta\|_{\dot{C}_t^\beta(L^\infty(\Omega))} B_{\theta_0}^{1+\tilde{\zeta}} \leq \frac{1}{2C}$ , then the second term on the RHS of (5.33) can be absorbed by the LHS, which gives the desired estimates.

Otherwise, if  $t^\beta \|\theta\|_{\dot{C}_t^\beta(L^\infty(\Omega))} B_{\theta_0}^{1+\tilde{\zeta}} > \frac{1}{2C}$ , then we perform time localization to adjust the time interval. Specifically, choose

$$\tau = \min\{T, \kappa (B_\theta(T)^{1+\tilde{\zeta}} \|\theta\|_{\dot{C}_T^\beta(L^\infty(\Omega))})^{-1/\beta}\}, \quad \kappa \ll 1.$$

Then for any  $t \in [0, \tau]$ , it holds that

$$(5.34) \quad \|(\partial_t u, \Delta u, \nabla P)\|_{L_t^p(L^r(\Omega))} \lesssim B_{\theta_0}^{1+\tilde{\zeta}} \|f\|_{L_t^p(L^r(\Omega))} + B_\theta^{(1+\tilde{\zeta})\cdot\frac{2q}{q-N}}(t) (B_\theta(t) - 1)^{r^*} \|u\|_{L_t^p(L^r(\Omega))},$$

where  $r^* = (1 + \tilde{\zeta}) \cdot \frac{2qr'}{q-N} + 2$ .

Next, we try to extend the above estimates to  $[0, T]$ . To this end, we perform a partition on time interval as the following:

Suppose  $\{\psi_k\}_{k \in \mathbb{N}}$  is a partition of unity of  $\mathbb{R}_+$  such that

$$\operatorname{supp} \psi_0 \subset [0, \tau], \quad \psi_0 \equiv 1 \text{ on } [0, \frac{\tau}{2}],$$

$$\text{supp } \psi_k \subset (\frac{k}{2}\tau, \frac{k}{2}\tau + \tau) \text{ for } k \geq 1, \quad \|\partial_t \psi_k\|_{L^\infty} \leq c\tau^{-1},$$

$$\sum_{k=0}^K \psi_k(t) = 1, \quad \forall t \in [0, T], \quad \frac{K}{2}\tau \leq T < \frac{K+1}{2}\tau.$$

Denote  $u_k = u\psi_k$ ,  $P_k = P\psi_k$ ,  $f_k = f\psi_k$ , then  $(u_k, P_k, f_k)$  satisfy the following system

$$(5.35) \quad \begin{cases} \partial_t u_k - \text{div}(\mu(\theta)\mathcal{D}(u_k)) + \nabla P_k = f_k + u\partial_t \psi_k, \\ \text{div } u_k = 0, \quad \int_{\Omega} P_k dx = 0, \\ u_k|_{t=\frac{k}{2}\tau} = 0, \quad u_k|_{\partial\Omega} = 0. \end{cases}$$

Let  $I_k = [\frac{k}{2}\tau, \frac{k}{2}\tau + \tau]$ ,  $k = 0, \dots, K-1$ ,  $I_K = [\frac{K}{2}\tau, T]$ .

For any  $t \in I_k$ , it follows from (5.34) that

$$(5.36) \quad \begin{aligned} & \|(\partial_t u_k, \Delta u_k, \nabla P_k)\|_{L^p(\frac{k}{2}\tau, t; L^r(\Omega))} \\ & \lesssim B_\theta^{1+\tilde{\zeta}}(t) (\|f_k\|_{L^p(\frac{k}{2}\tau, t; L^r(\Omega))} + \|u\partial_t \psi_k\|_{L^p(\frac{k}{2}\tau, t; L^r(\Omega))}) \\ & \quad + B_\theta^{(1+\tilde{\zeta})\frac{2q}{q-N}}(t) (B_\theta(t) - 1)^{r^*} \|u_k\|_{L^p(\frac{k}{2}\tau, t; L^r(\Omega))}. \end{aligned}$$

Notice that

$$(5.37) \quad \begin{aligned} & \|u\partial_t \psi_k\|_{L^p(\frac{k}{2}\tau, t; L^r(\Omega))} \lesssim \tau^{-1} \|u\|_{L^p(\frac{k}{2}\tau, t; L^r(\Omega))} \\ & \lesssim (B_\theta(T)^{1+\tilde{\zeta}} \|\theta\|_{\dot{C}_T^\beta(L^\infty)})^{1/\beta} \|u\|_{L^p(\frac{k}{2}\tau, t; L^r(\Omega))}. \end{aligned}$$

Substituting (5.37) into (5.36), then summing up over  $k$  gives that

$$(5.38) \quad \begin{aligned} & \|(\partial_t u, \Delta u, \nabla P)\|_{L_T^p(L^r(\Omega))} \lesssim B_\theta^{1+\tilde{\zeta}}(T) \|f\|_{L_T^p(L^r(\Omega))} \\ & \quad + (B_\theta^{(1+\tilde{\zeta})\frac{2q}{q-N}}(T) (B_\theta(T) - 1)^{r^*} + B_\theta(T)^{(1+\tilde{\zeta})(1+\frac{1}{\beta})} \|\theta\|_{\dot{C}_T^\beta(L^\infty)}^{1/\beta}) \|u\|_{L_T^p(L^r(\Omega))}. \end{aligned}$$

By the definition of  $C_\theta(t)$ , (5.30) is proved. We remark that  $l_2 \geq \min\{r^* \frac{q}{q-N}, \frac{1}{\beta}\} > 1$ . □

(c) *Existence and uniqueness of solution to (2.5) with null initial data*

**Theorem 5.6.** *Suppose all the assumptions in Theorem 2.3 are true and  $u_0 = 0$ , then the system (2.5) has a unique strong solution  $(u, P)$  on  $\Omega_T$  satisfying*

$$(5.39) \quad \|(\partial_t u, \Delta u, \nabla P)\|_{L_t^p(L^r(\Omega))} \leq C B_\theta^{1+\tilde{\zeta}}(t) \exp(Ct C_\theta(t)) \|f\|_{L_t^p(L^r(\Omega))},$$

for any  $t < T$ , where  $B_\theta(t)$ ,  $C_\theta(t)$  and  $C$  are defined as in Theorem 2.3.

*Proof.* The proof of existence of local strong solution is trivial after *a priori* estimates (5.39). Thus we only give the proof of this estimates.

Suppose  $(u, P)$  is a smooth solution on  $\Omega_T$ , indeed, we have

$$(5.40) \quad \frac{d}{dt} \|u\|_{L^r(\Omega)} \leq \|\partial_t u\|_{L^r(\Omega)}.$$

Taking advantage of (5.38) and (5.40), then for all  $\epsilon > 0$

$$\begin{aligned}
(5.41) \quad \|u(t, \cdot)\|_{L^r(\Omega)}^p &= p \int_0^t \|u(\tau, \cdot)\|_{L^r(\Omega)}^{p-1} \frac{d}{d\tau} \|u(\tau, \cdot)\|_{L^r(\Omega)} d\tau \\
&\leq (p-1)\epsilon \int_0^t \|u(\tau, \cdot)\|_{L^r(\Omega)}^p d\tau + \epsilon^{1-p} \int_0^t \|\partial_t u\|_{L^r(\Omega)}^p d\tau \\
&\leq ((p-1)\epsilon + \epsilon^{1-p} C C_\theta^p(t)) \|u\|_{L_t^p(L^r(\Omega))}^p \\
&\quad + \epsilon^{1-p} C B_\theta^{(1+\tilde{\zeta})p}(t) \|f\|_{L_t^p(L^r(\Omega))}^p.
\end{aligned}$$

Now, let us denote

$$\begin{aligned}
\alpha(t) &= \epsilon + \epsilon^{1-p} C C_\theta^p(t), \quad \beta(t) = \epsilon^{1-p} B_\theta^{(1+\tilde{\zeta})p}(t) \|f\|_{L_t^p(L^r(\Omega))}^p, \\
F(t) &= \|u\|_{L_t^p(L^r(\Omega))}^p, \quad \gamma(t) = \int_0^t \beta(s) ds,
\end{aligned}$$

then from (5.41), it follows that

$$(5.42) \quad F(t) \leq C \int_0^t \alpha(s) F(s) ds + C \gamma(t).$$

Therefore, Gronwall's lemma implies

$$\begin{aligned}
(5.43) \quad F(t) &\leq C \gamma(t) \exp(C \int_0^t \alpha(s) ds) \\
&\leq C \epsilon^{1-p} t B_\theta^{(1+\tilde{\zeta})p}(t) \|f\|_{L_t^p(L^r(\Omega))}^p \exp(C \epsilon t + C \epsilon^{1-p} t C_\theta^p(t)).
\end{aligned}$$

Choosing  $\epsilon = C_\theta(t)$ , then (5.43) turns into

$$(5.44) \quad \int_0^t \|u(\tau, \cdot)\|_{L^r(\Omega)}^p d\tau \lesssim C_\theta^{-p}(t) B_\theta^{(1+\tilde{\zeta})p}(t) \|f\|_{L_t^p(L^r(\Omega))}^p \exp(C t C_\theta(t)),$$

where we have used  $t C_\theta(t) \lesssim \exp(t C_\theta(t))$ .

Inserting (5.44) into (5.30), we finally obtain (5.39).  $\square$

**5.0.2. General initial data.** In this section, we generalize the previous result to the case of general initial data. Consider the following two systems:

$$(5.45) \quad \begin{cases} \partial_t \omega - \mu(\underline{\theta}) \Delta \omega + \nabla \Pi = f, \\ \operatorname{div} \omega = 0, \quad \int_\Omega \Pi dx = 0, \\ \omega|_{t=0} = u_0, \quad \omega|_{\partial\Omega} = 0, \end{cases} \quad \begin{cases} \partial_t v - \operatorname{div}(\mu(\theta) \mathcal{D}(v)) + \nabla Q \\ = (\mu(\theta) - \mu(\underline{\theta})) \Delta \omega + \mu'(\theta) \nabla \theta \mathcal{D}(\omega), \\ \operatorname{div} v = 0, \quad \int_\Omega Q dx = 0, \\ v|_{t=0} = 0, \quad v|_{\partial\Omega} = 0, \end{cases}$$

where  $\underline{\theta} = \inf_{x \in \Omega} \theta_0$ . It's easy to verify that  $u = v + \omega$  and  $P = \Pi + Q$  satisfy system (2.5) if  $\omega$  and  $v$  satisfy the corresponding system.

Theorem 5.1 implies that there exists a unique solution  $(\omega, \Pi)$  to the first system of (5.45) such that for any  $0 < t < T$

$$(5.46) \quad \|(\partial_t \omega, \Delta \omega, \nabla \Pi)\|_{L_t^p(L^r(\Omega))} + \|\omega\|_{L_t^\infty(D_{A_r}^{1-\frac{1}{p}, p})} \lesssim \|f\|_{L_t^p(L^r(\Omega))} + \|u_0\|_{D_{A_r}^{1-\frac{1}{p}, p}}.$$

On the other hand, Theorem 5.6 implies that there exists a unique solution  $(v, Q)$  to the second system of (5.45) such that

$$\begin{aligned}
 & \|(\partial_t v, \Delta v, \nabla Q)\|_{L_t^p(L^r(\Omega))} + \|v\|_{L_t^\infty(D_{A_r}^{1-\frac{1}{p}, p})} \\
 (5.47) \quad & \lesssim B_\theta^{1+\tilde{\zeta}}(t) \exp(CtC_\theta(t)) (\|(\mu(\theta) - \mu(\underline{\theta}))\Delta\omega\|_{L_t^p(L^r(\Omega))} + \|\nabla\theta\nabla\omega\|_{L_t^p(L^r(\Omega))}) \\
 & \lesssim B_\theta^{2+\tilde{\zeta}}(t) \exp(CtC_\theta(t)) (\|f\|_{L_t^p(L^r(\Omega))} + \|u_0\|_{D_{A_r}^{1-\frac{1}{p}, p}}),
 \end{aligned}$$

where we have used (5.46).

Adding up (5.46) and (5.47) yields that

$$\begin{aligned}
 & \|u\|_{L_t^\infty(D_{A_r}^{1-\frac{1}{p}, p})} + \|(\partial_t u, \Delta u, \nabla P)\|_{L_t^p(L^r(\Omega))} \\
 (5.48) \quad & \lesssim B_\theta^{2+\tilde{\zeta}}(t) \exp(CtC_\theta(t)) (\|u_0\|_{D_{A_r}^{1-\frac{1}{p}, p}} + \|f\|_{L_t^p(L^r(\Omega))}).
 \end{aligned}$$

Thus, (2.6) is proved.

To prove (2.7), we apply Theorem 5.5 to the second system of (5.45),

$$\begin{aligned}
 & \|(\partial_t v, \Delta v, \nabla v)\|_{L_t^p(L^r(\Omega))} \\
 & \lesssim B_\theta^{1+\tilde{\zeta}}(t) \left( \|(\mu(\theta) - \mu(\underline{\theta}))\Delta\omega\|_{L_t^p(L^r(\Omega))} + \|\nabla\theta\nabla\omega\|_{L_t^p(L^r(\Omega))} \right) \\
 (5.49) \quad & + C_\theta(t) \|v\|_{L_t^p(L^r(\Omega))} \\
 & \lesssim B_\theta^{2+\tilde{\zeta}}(t) (\|f\|_{L_t^p(L^r(\Omega))} + \|u_0\|_{D_{A_r}^{1-\frac{1}{p}, p}}) + C_\theta(t) \|v\|_{L_t^p(L^r(\Omega))}.
 \end{aligned}$$

Summing up (5.46) and (5.49) and noticing that

$$\|v\|_{L_t^p(L^r(\Omega))} \leq \|u\|_{L_t^p(L^r(\Omega))} + \|\omega\|_{L_t^p(L^r(\Omega))},$$

we finally obtain (2.7). This completes the proof of Theorem 2.3.

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